

MANAGING OPERATIONS UNDER BANKRUPTCY RISK

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This dissertation focuses on the relationship between a firm's operational decisions and its bankruptcy risk. It consists of three self-contained chapters. All three chapters are joint work with Professor Vishal Gaur.

Chapter 1 studies the implications of asset based lending for operational investment, probability of bankruptcy, and capital structure for a borrower firm. We set up a single-period game with two players, a business owner and a bank. The business owner decides how to allocate her capital between the equity of a new business and the external capital market in order to maximize her expected profit. We model the new business as a single-period inventory (newsvendor) model. The bank does not know the newsvendor's demand distribution, and sets an asset based credit limit to maximize its expected profits. We show that the equilibrium order quantity is a function of market parameters, and deviates from the classical newsvendor solution. In this solution, asset based lending leads to an upper limit on the potential loss faced by the bank, and thus, helps manage bankruptcy risk. In particular, the collateral value of inventory is a function of the bank's belief regarding the firm's demand distribution because the amount of inventory that will have to be liquidated in case of a default is random and depends on the realized demand. We also show that the probability of bankruptcy and the capital structure at equilibrium are functions of information asymmetry, bankruptcy costs, and the newsvendor model parameters.

Chapter 2 focuses on a cash constrained firm that has to balance growth

and bankruptcy risk when making its operational and borrowing decisions. We study the operational implications of this tradeoff by setting up a finite horizon cash-constrained inventory model with non-stationary demand, which is a function of the firm's past sales. We analyze four different growth scenarios: unconstrained growth, self-financing growth, growth under reorganization bankruptcy, and growth under liquidation bankruptcy. These scenarios capture different aspects of the impact of financing constraints and the bankruptcy process on a firm's operational decisions. Our analysis shows that a self-financing growth strategy, which avoids risky borrowing, is overly conservative and that growth requires making risky operational investment decisions. That is, a cash constrained firm should take some risk and over-invest (i.e., order more than the classical single period newsvendor quantity) to fuel growth. However, the firm should be cautious because over-investment amplifies bankruptcy risk. Hence, we show that the firm needs to achieve the right balance between growth and bankruptcy risk to maximize its long term profits.

Chapter 3 investigates whether inventory productivity explains financial distress for retailers. Inventory is a key management item, which usually is the largest current asset in a retailer's books. Since a vast majority of a retailer's operational decisions are related to inventory, we hypothesize that retailers with high inventory productivity have lower probability of bankruptcy. Using a data set of retail bankruptcies, we test this hypothesis by adding inventory turnover as an explanatory variable to three commonly used bankruptcy prediction models. Our analysis shows that inventory turnover significantly improves the model fit in all three models and that retailers with high inventory turnover have lower probability of bankruptcy. These results have important implications for bankruptcy prediction and turnaround management.

BIOGRAPHICAL SKETCH

Yasin Alan was born in Eskişehir, a beautiful city in northwestern Turkey. He started talking when he was six months old, and has not stopped being loud since then.

Yasin spent much of his early childhood with his grandparents in Eskişehir. His parents moved occasionally. He finished primary school in Kızılcahamam, Ankara, middle school in Polatlı, Ankara, and high school in Balıkesir. After graduating from Balıkesir Science High School with unforgettable memories, he spent one year at Boğaziçi University in İstanbul trying to learn how to be loud in English.

Yasin moved to the United States when he was 19. He lived in College Station, Texas, for four years, and has been residing in gorge(ou)s Ithaca, New York, since 2006. He holds a bachelor's degree in industrial and systems engineering from Texas A&M University and two master's degrees, one in operations research and the other in operations management, from Cornell University. Upon completion of his doctoral studies at the Johnson Graduate School of Management at Cornell, he will join the Owen Graduate School of Management at Vanderbilt University as an assistant professor of operations management.

Yasin is an avid Galatasaray fan.

To my grandparents, Hatice and Hasan Hüseyin Karakaş

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CHAPTER 1

OPERATIONAL INVESTMENT AND CAPITAL STRUCTURE UNDER ASSET BASED LENDING: A ONE PERIOD MODEL

1.1 Introduction

Asset Based Lending (ABL) is a method commonly used by banks to lend money to small and risky businesses. In this method, a bank sets a maximum lending amount secured by the current assets of the borrowing firm, which include its inventory, cash, and account receivables. Inventory is an important basis for ABL; asset based loans secured solely by inventory are common in practice, especially in the retail industry (GE Capital [36], page 14), which is one of the top three asset based borrowers (Commercial Finance Association [25]). Banks use simple rules of thumb to assess the collateral value of inventory, such as its type (i.e., raw material, work in process, finished goods) and its age (GE Capital [36]). However, such rules of thumb do not optimize the credit limit or the collateral value of inventory when the demand faced by the firm is uncertain.

ABL is a large industry. In 2009, the total amount of outstanding asset based loans in the USA was \$480 billion (Commercial Finance Association [25]), which constituted 25% of the total amount of loans and short term papers issued to nonfinancial corporations (the Federal Reserve [12]). ABL is useful to banks because they are usually not well-informed regarding the future demand prospects of the borrower. ABL mitigates the cost of this information asymmetry by preventing over-borrowing. Furthermore, since an asset based loan is secured by the borrower's current assets, the bank can recover some of its

losses by liquidating the borrower's assets in case of a default. ABL is also useful to small businesses by providing them access to low interest rate financing. Such businesses typically do not have access to cash flow financing, which is availed by large companies with revenues in excess of \$25 million and stable profits (Burroughs [14]). Examples of such low interest rate financing include programs offered by the Small Business Association (SBA), such as the Standard Asset Based CAPLines program (Godfrey [37]) and the Small Business Lending Fund established for community banks as a part of the Small Business Jobs Act (The Secured Lender Industry News [80]). These programs allow participating banks to provide asset based loans to small businesses, wherein the SBA sets a maximum allowable interest rate. For example, this interest rate was equal to the prime rate plus 2.25% as of November 2011 (SBA [74]).

This chapter studies the implications of ABL for the operational investment, capital structure, and probability of bankruptcy of a borrower firm. We set up a single-period game theoretic model with two players, a business owner and a bank. The business owner decides how to allocate her capital between the equity of a new business and the external capital market in order to maximize her expected profit. The business is represented using the newsvendor model. The owner sets up the business as a limited liability firm, interacts with the bank on its behalf, and manages its operations. The bank provides a loan to the firm at a fixed interest rate. It sets a credit limit, secured by inventory, to maximize its expected profit when it is partially informed about the demand distribution of the newsvendor firm.

We address three research questions. First, what should be the equity investment and order quantity decisions of the owner at equilibrium? Second,

how should the bank determine the collateral value of inventory and the asset based credit limit to maximize its expected profit under information asymmetry? Third, what is the probability of bankruptcy of the firm at equilibrium? Through these questions, we show that firm characteristics, captured by the newsvendor model parameters, and the economic environment, captured by the return on the external investment alternative, the lending interest rate, and the lender's sentiment (i.e., the bank's belief regarding the newsvendor's demand distribution), affect the owner's operational and financial decisions, the credit limit offered by the bank, and the probability of bankruptcy.

One result of our study is that the owner's choice of debt and equity shapes the firm's order quantity through the overage and underage costs of the newsvendor model. This choice is driven by the tradeoff between investing in the external market and investing in the newsvendor. For example, if the external capital market offers a higher return, then the owner chooses to invest a smaller amount in the equity of the firm and rely more on debt to finance the inventory. However, the bank prevents the firm from over-borrowing by imposing an asset based credit limit. As a result, the equilibrium outcome can be of three types: when the firm borrows and has non-zero probability of bankruptcy, when the firm borrows with zero probability of bankruptcy, and when the firm does not borrow.

A second result of our study is the structure of the optimal asset based credit and the collateral value of inventory. The credit limit decomposes into two components: a riskless component which can be recovered even if the realized demand is zero, and a risky component which is tied to the firm's demand prospects and information asymmetry. Thus, the bank's belief regarding the

firm's demand distribution plays an important role. For example, when the bank is pessimistic about the demand prospects of the firm, it lowers the collateral value of inventory to tighten the credit limit. However, a tight credit limit might lead to a counter-intuitive equilibrium outcome in which the owner increases the order quantity by injecting more equity in the firm. Moreover, the collateral value of inventory becomes a function of the bank's belief regarding the firm's demand distribution and other parameters. This result contrasts with the common practice of banks to use simple rules of thumb to value inventory and set a credit limit.

The third main result of our study is related to the equilibrium probability of bankruptcy. Interestingly, we find that when the firm borrows with bankruptcy risk, the credit limit is *always* binding. In other words, a solution in which the firm may borrow with bankruptcy risk and not use up the entire credit limit never arises at equilibrium. Furthermore, we find that the bank sets the optimal credit limit in such a way that the probability of bankruptcy becomes independent of the owner's choice of debt and equity as long as the firm borrows with risk. We also show that the probability of bankruptcy depends on the news vendor parameters. For example, it increases in the salvage value of the inventory because the credit limit and the equilibrium order quantity both increase in the salvage value.

1.2 Literature Review

Despite its practical usage, ABL has not been well studied in the academic literature. The majority of the analytical models of capital structure in corporate

finance focus on bondholders, and do not include the setting of a credit limit. Further, most of these models do not capture the details of the operational decisions. In the operations literature, ABL has been analyzed by Buzacott and Zhang [15]. While their paper provides many insights, it treats the equity of the firm as given, and so does not model the implication of ABL for capital structure. Moreover, there is no information asymmetry. Therefore, our work contributes to the operations-finance interface literature by modeling the equity, borrowing, and operational investment decisions of the business owner in the presence of an asset based credit limit, information asymmetry, taxation, and costly bankruptcy. The literature in these areas being vast, we describe relevant papers in brief.

Modigliani and Miller [61] show that, in a perfect market, the capital structure of a firm is irrelevant to its optimal operational decisions. That is, the decision that maximizes the value to shareholders is equal to the decision that maximizes the total value of the firm. Subsequent research has led to two competing theories of capital structure, the tradeoff theory and the pecking order theory. Research on the tradeoff theory shows the existence of an optimal capital structure due to market frictions such as interest rate spread, taxation, costly bankruptcy, and liquidity constraints (e.g., Modigliani and Miller [62], Kraus and Litzenberge [48], Gordon [38]), but under *complete information*. A few papers in this stream consider firms' operational decision models in detail. In particular, Stiglitz [76] shows a connection between operational and financial decisions under bankruptcy risk, and Dotan and Ravid [28] model the optimal capacity, financing, and production decisions with uncertain sales price. In contrast to the tradeoff theory, the pecking order theory shows the existence of a financing hierarchy that minimizes the costs related to *incomplete information* (Jensen and

Meckling [45], Myers [66], Myers and Majluf [65], and Childs et al. [22]). According to Frank and Goyal [35], the pecking order theory models are relatively simple with linear objective functions, and thus, illustrate financing hierarchy under strong modeling assumptions, rather than giving a unifying framework.

The recent literature in capital structure is largely empirical. Interestingly, it obtains some findings that relate to operational characteristics of firms but cannot be explained by the theoretical models. For example, Lemmon et al. [55] show that there is substantial unexplained variation in capital structure, which is firm-specific and time-invariant; Rauh and Sufi [72] illustrate that what a firm produces and the production assets it uses are the most important determinants of capital structure in the cross section; and Campello and Giambona [18] show that firms with redeployable assets have more debt capacity. Other empirical papers have investigated the relationship of capital structure with various firm characteristics, including profitability, growth, liquidation value, return volatility, and operational risks. See Harris and Raviv [40] and Leary and Graham [51] for extensive reviews of the empirical capital structure literature.

Our study builds on the capital structure literature by studying ABL under information asymmetry, incorporating the newsvendor model framework, and allowing an external investment option to the owner. This combination of a practical borrowing model and a realistic investment scenario reveals new insights regarding the interaction of the owner's operational and financial decisions. For example, our model yields predictions that are consistent with empirical observations in corporate finance with respect to the impact of operational characteristics, such as profitability, demand volatility, and asset recoverability (salvage value in our model), on a firm's capital structure. Additionally, con-

trary to the majority of the asymmetric information models that lead to underinvestment (Hubbard [44]), our study shows a non-monotone relationship between information asymmetry and operational investment.

The operations management literature on joint operational-financial decisions addresses market imperfections by including taxes, liquidity constraints, bankruptcy risk, costly issuance of debt and equity, and credit limits into single- and multi-period inventory models. Among single-period models, Xu and Birge [82] investigate the tradeoff between bankruptcy costs and the tax benefits of debt in a cash-constrained newsvendor model. Their analysis shows that integrating operational and financial decisions can improve firm value. Buzacott and Zhang [15] study single- and multi-period models with asset based credit limit. The second half of their paper is relevant to our work. It analyzes a single-period model in which a newsvendor and a bank seek to maximize own profits in the presence of bankruptcy risk. Dada and Hu [26] use a similar framework, with the difference that the bank chooses an optimal interest rate to charge to the newsvendor, instead of imposing a borrowing limit. They show the existence and uniqueness of an equilibrium order quantity-interest rate pair. Multi-period inventory models analyze similar operational issues with additional financial dynamics, such as cash flows, dividend payments and capital subscriptions, e.g., Li et al. [56], Hu and Sobel [42], Chao et al. [20], and Hu et al. [43].

Research on the impact of financial considerations on operational decisions is not limited to inventory models. Financial constraints and the risk of bankruptcy also affect the firm's survival strategy (Archibald et al. [6]), relations with its supply chain partners (Lai et al. [50], Babich [8], Kouvelis and Zhao [47], and Yang and Birge [84]), the choice of production technologies (Led-

erer and Singhal [53], and Boyabatli and Toktay [13]), the optimal time to shut down a firm (Xu and Birge [83]), and the optimal time to offer an IPO (Babich and Sobel [9]).

We contribute to the single-period operations management models by endogenizing the owner's equity decision. In our model, the owner faces a trade-off between investing in the newsvendor and investing in the external capital market, which shapes the equity investment decision and puts the firm's operational decisions in a broader context. Furthermore, we introduce the bank as a second player in the model in order to incorporate information asymmetry. Thus, the owner is not the only decision maker. This setting captures the impact of market conditions (e.g., the external market return, the bank's lending sentiment) on the equilibrium order quantity, the capital structure, and the risk of bankruptcy.

1.3 Model

We set up a single-period model with two players, a small business owner and a commercial bank. The owner has capital K to invest and wishes to allocate this capital between a new business and the external capital market. The new business is a single-period inventory (newsvendor) model. The owner makes three related decisions: how much equity and debt to have in the newsvendor business and what stocking quantity to purchase. The first two constitute the capital structure of the firm, and the third its operational investment. Let $x \in [0, K]$ denote the amount of equity of the newsvendor, w its borrowing amount, and q its stocking quantity.

The owner constitutes the newsvendor business as a limited liability firm. This means that the owner and the firm are separate legal personalities, as per corporate law, and the owner's loss is limited to the amount of equity in case the firm defaults on the loan. However, since the owner makes the stocking and borrowing decisions for the firm, their objectives are aligned. We use the terms newsvendor and firm interchangeably.

The bank is a monopoly that seeks to maximize its expected profit by lending to the firm. We assume that the bank does not know the true demand distribution of the firm. It charges a fixed interest rate α , and sets an asset-based credit limit to prevent over-borrowing. Assuming a fixed interest rate, instead of allowing the bank to optimize the interest rate or set it to achieve a risk-free rate of return under its belief of the demand distribution, helps simplify our model and focus on the implications of asset based lending. This assumption is also practically relevant for small businesses that obtain asset based loans under programs governed by the SBA. As we discussed in Section 1.1, the SBA sets a maximum interest rate and many small businesses borrow at that rate.¹ Finally, increasing the interest rate can lower the bank's profit by changing the set of borrowing firms and inducing firms to make riskier investments (Stiglitz and Weiss [77]). The literature on this well known adverse selection problem implies that the lending interest rate should be optimized for a population of borrowers, not for an individual firm; see Greenwald and Stiglitz [39] (page 15).

¹Bates [10] (page 230) discusses examples of small firms that borrow at the maximum interest rate set by the SBA, e.g., "Exim Capital [an asset based lender], [is] typically charging its borrowers interest rates in the 15% annual range... Chun [a manager at Exim Capital] would like to charge 18%, but SBA regulations hold him currently to 15%...[T]he collateral [is] involved in six typical loans made by Exim Capital, which involved secured loans to (a) Jee and Jung Cleaners, (b) C.H.A. Kyung, Inc., (c) Tri J. Tires, (d) H.S.K. Cleaners, (e) 104 Broadway Farm (a grocery), and (f) Chonel, Inc. Loan sizes ranged from \$52,000 to \$105,000 in these six transactions...[T]he six deals offered collateral to Exim Capital ranging from \$162,500 to \$801,000." Lending interest rates are also closely monitored and capped in developing countries with a history of loan-sharking (Park [69]).

It also implies that a credit allocation mechanism, such as asset based lending, mitigates adverse selection by preventing over-borrowing. In Section 1.6, we discuss these issues in detail, present alternative models with interest rate optimization, and illustrate that using ABL leads to a higher expected profit for the bank than optimizing the interest rate without a credit limit.

Our remaining assumptions are as follows. Similar to the capital structure literature, which deals with a tradeoff between the tax shield of debt and bankruptcy costs, we assume that the firm pays taxes on its income at a corporate tax rate τ , and the after-tax income accrues to the owner. We also assume that the bank incurs a bankruptcy cost proportional to its loss if the firm defaults. We do not apply corporate taxes to the bank's income or personal taxes to the owner because there are no economic tradeoffs related to these taxes in our model. Taxes will merely scale the respective incomes without changing the nature of results. Therefore, we ignore them for simplicity, and assume that the bank and the owner live for more than one period and have other sources of income that they can offset losses against.

The sequence of events is as follows. First, the owner determines the amount of equity of her firm, $x \in [0, K]$. Then, she manages the firm. She interacts with the bank on behalf of the firm and borrows w without exceeding a credit limit ψ set by the bank. This gives her a total capital of $x + w$ to run the newsvendor business. She orders q units by paying cq to her supplier, where c denotes the per unit cost. The amount cq cannot exceed $x + w$ because the payment to the supplier is due when the order is placed. A random demand ξ occurs. If ξ is small enough so that the firm cannot repay the loan with interest, then the firm declares bankruptcy and its cash and inventory are possessed by the bank.

Otherwise, the firm generates a revenue of $p \min\{\xi, q\} + s(q - \xi)^+$, where p denotes the selling price, s denotes the salvage value of unsold units, and $(q - \xi)^+ \equiv \max\{q - \xi, 0\}$. The pre-tax *operating income* of the firm is thus equal to $p \min\{\xi, q\} + s(q - \xi)^+ - cq - \alpha w$. The firm pays corporate tax at rate τ if the operating income is positive, and zero tax if it incurs an operating loss. It repays the loan plus interest, $(1 + \alpha)w$, to the bank and its after-tax *ending cash position* accrues to the owner.

Since we investigate asset based lending, the inventory of the firm must have a positive salvage value so that it can be used as collateral. Hence, we have $s > 0$. A good illustration of the usefulness of salvaging inventory in ABL is provided by the bankruptcy filing and the subsequent inventory liquidation of the Borders Group, Inc. Borders obtained an asset based loan of \$700 million in April 2010 from a consortium of lenders (ABF Journal [7]). After its Chapter 11 bankruptcy protection in February 2011 and Chapter 7 liquidation in July 2011, Gordon Brothers Group and Hilco Merchant Resources sold Borders' inventories at 40-60 percent discounts (Legal News [54]).

The demand ξ is non-negative and follows a continuous probability distribution with increasing failure rate (IFR). The pdf, cdf, complementary cdf (ccdf), and inverse ccdf of the demand distribution are denoted as f , F , \bar{F} , and \bar{F}^{-1} , respectively, where f is positive on an interval and zero elsewhere. Also let g , G , \bar{G} denote the pdf, cdf, and ccdf, respectively, of the bank's belief of the demand distribution. The bank uses this belief to assess the newsvendor's loan application. All parameters other than the newsvendor's true demand distribution are common knowledge. For the newsvendor problem to be non-trivial, we assume that $(1 - \tau)(p - s) > (1 + \alpha)c - s$ and $c > s$. These assumptions are necessary for

the tail probability of the demand distribution to lie between 0 and 1. We also require that $\frac{p-s}{(1+\alpha)c-s} \geq 1 + \alpha$, i.e., the profit margin of the newsvendor is sufficiently high so that the rate of return of a sold unit that is purchased on credit, $\frac{p-s}{(1+\alpha)c-s} - 1$, is no less than the borrowing rate α .

Our analysis proceeds in the reverse sequence of time. We first solve for the newsvendor's stocking decision and the bank's credit limit decision given the equity of the newsvendor. We then solve the newsvendor-bank sub-game and the owner's capital allocation problem, which determines her firm's capital structure and order quantity at equilibrium.

1.3.1 The Newsvendor's Problem

The newsvendor's ending cash position after payment of loan with interest and taxes is

$$\begin{aligned} \pi(q, w, x, \xi) = & \left(x + w - cq + p \min\{\xi, q\} + s(q - \xi)^+ - (1 + \alpha)w \right. \\ & \left. - \tau [p \min\{\xi, q\} + s(q - \xi)^+ - cq - \alpha w]^+ \right)^+. \end{aligned}$$

Thus, the owner solves the following problem for the newsvendor as a function of the equity x in order to determine her payoff:

$$\pi^*(x) = \max_{(q, w) \in \mathbb{C}(x, \psi)} E_F [\pi(q, w, x, \xi)]. \quad (1.1)$$

Here, E_F denotes expectation with respect to the distribution of demand, and $\mathbb{C}(x, \psi) = \{(q, w) \in \mathbb{R}_+^2 : w \leq \psi, cq \leq x + w\}$ is the constraint set, where \mathbb{R} denotes the real line. The first constraint is that the borrowing amount w must be less than the credit limit ψ set by the bank. The second constraint specifies that the cost of procurement must be less than the total cash available to the newsvendor at the

outset. We solve the newsvendor's problem ignoring the first constraint. Then, we solve the bank's problem to compute ψ . Finally, we determine conditions in which the credit limit is binding.

It is important to note that the newsvendor will not simultaneously borrow and hold excess cash, i.e.,

$$w = (cq - x)^+. \quad (1.2)$$

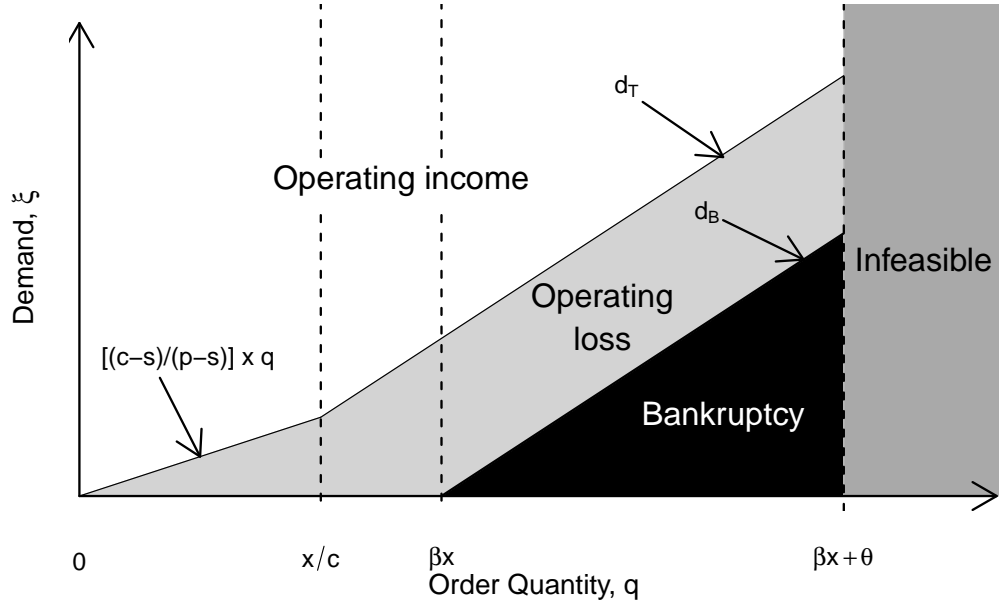
Intuitively, this condition occurs because $\alpha > 0$. We omit the proof, which follows by a contradiction argument. This condition implies that there are three possible scenarios that can occur based on the values of the starting capital and the order quantity of the newsvendor. We denote as (NB) the scenario in which the newsvendor has enough cash to purchase inventory and does not borrow any money from the bank. In scenario (BWO), the newsvendor's borrowing amount is sufficiently small as to be without bankruptcy risk. In scenario (BWR), the newsvendor borrows with bankruptcy risk. We use the superscripts NB, BWO and BWR to denote variables in the respective solutions. Figure 1.1 depicts the regions defining these three scenarios as functions of the order quantity q , the demand realization ξ , and the equity x .

We now characterize the newsvendor's expected ending cash position under each scenario. Scenario (NB) occurs when $q \leq x/c$. In this scenario, the newsvendor has operating income and pays tax if $\xi \geq \frac{c-s}{p-s}q$. It has operating loss if $\xi < \frac{c-s}{p-s}q$. Thus, its ending cash position is

$$\pi^{NB}(q, 0, x, \xi) = x + p \min\{\xi, q\} + s(q - \xi)^+ - cq - \tau(p - s) \left[\min\{\xi, q\} - \frac{c-s}{p-s}q \right]^+.$$

By taking an expectation, we obtain the newsvendor's expected ending cash

Figure 1.1: Three cases that can arise as a function of x , q , and ξ . (NB): The newsvendor does not borrow if $q \leq x/c$. It has operating income if $\xi > \frac{c-s}{p-s}q$. Otherwise, it has operating loss. (BWO): The newsvendor borrows without risk if $x/c < q \leq \beta x$. It has operating income if $\xi > d_T = \frac{[(1+\alpha)c-s]q-\alpha x}{p-s}$. Otherwise, it has operating loss. (BWR): The newsvendor borrows with risk if $\beta x < q \leq \beta x + \theta$. It has operating income if $\xi > d_T$. It has operating loss if $\xi < d_T$. It declares bankruptcy if $\xi < d_B = \frac{[(1+\alpha)c-s]q-(1+\alpha)x}{p-s}$. It cannot order more than $\beta x + \theta$ due to the asset based credit limit imposed by the bank. The values of β and θ are defined in Proposition 1.3.3.



position in (NB) as

$$E_F[\pi^{NB}] = x + (p-s) \left(\int_0^q \bar{F}(\xi) d\xi - \tau \int_{\frac{c-s}{p-s}q}^q \bar{F}(\xi) d\xi \right) - (c-s)q. \quad (1.3)$$

Scenario (BWO) occurs when the order quantity is greater than x/c , but is sufficiently small so that the newsvendor does not default even when realized demand is zero. For $q > x/c$, the newsvendor's ending cash position when it realizes demand $\xi = 0$ is given by:

$$(1+\alpha)x - [(1+\alpha)c-s]q - \tau(\alpha x - [(1+\alpha)c-s]q)^+. \quad (1.4)$$

Observe that $(\alpha x - [(1+\alpha)c-s]q)^+ = 0$ for any $q > x/c$. That is, the newsvendor

has operating loss, which implies that q must not exceed $\frac{1+\alpha}{(1+\alpha)c-s}x$ to ensure that (1.4) is non-negative. Thus, in scenario (BWO), $x/c < q \leq \frac{1+\alpha}{(1+\alpha)c-s}x$. Further, to compute the newsvendor's ending cash position, we note that the newsvendor has taxable income if $\xi \geq d_T(q, x) \equiv \frac{(1+\alpha)c-s}{p-s}q - \frac{\alpha}{p-s}x$, and loss otherwise. Therefore, the newsvendor's ending cash position in (BWO) is

$$\begin{aligned}\pi^{BWO}(q, w, x, \xi) &= (1 + \alpha)(x - cq) + p \min\{\xi, q\} + s(q - \xi)^+ \\ &\quad - \tau(p - s) [\min\{\xi, q\} - d_T(q, x)]^+.\end{aligned}$$

Taking an expectation gives

$$E_F[\pi^{BWO}] = (1 + \alpha)x + (p - s) \left(\int_0^q \bar{F}(\xi) d\xi - \tau \int_{d_T(q, x)}^q \bar{F}(\xi) d\xi \right) - [(1 + \alpha)c - s]q. \quad (1.5)$$

Scenario (BWR) occurs and the newsvendor borrows with bankruptcy risk if $q > \frac{1+\alpha}{(1+\alpha)c-s}x$. For each such value of q , let $d_B(q, x)$ denote the demand realization below which bankruptcy occurs. The ending cash position at $\xi = d_B(q, x)$ is

$$\begin{aligned}\pi(q, w, x, d_B(q, x)) &= (1 + \alpha)x + (p - s)d_B(q, x) - [(1 + \alpha)c - s]q \\ &\quad - \tau(\alpha x + (p - s)d_B(q, x) - [(1 + \alpha)c - s]q)^+.\end{aligned}$$

Observe that when the firm is bankrupt, its operating income cannot be positive and there are zero taxes. Using this and setting $\pi(q, w, x, d_B(q, x))$ equal to zero gives the bankruptcy threshold $d_B(q, x) = \left(\frac{(1+\alpha)c-s}{p-s}q - \frac{1+\alpha}{p-s}x \right)^+$. For $d_B(q, x) \leq \xi \leq d_T(q, x)$, the newsvendor survives, but has an operating loss. The newsvendor earns an operating income if demand is higher than the threshold $d_T(q, x)$. Thus, we get

$$\pi^{BWR}(q, w, x, \xi) = \begin{cases} 0 & \text{if } \xi \leq d_B, \\ (p - s) [\min\{\xi, q\} - d_B(q, x)] & \text{if } d_B < \xi \leq d_T, \\ (p - s) [d_T(q, x) - d_B(q, x)] \\ \quad + (1 - \tau)(p - s) [\min\{\xi, q\} - d_T(q, x)] & \text{if } \xi > d_T. \end{cases}$$

Taking an expectation gives

$$E_F [\pi^{BWR}] = (p - s) \left[\int_{d_B(q,x)}^q \bar{F}(\xi) d\xi - \tau \int_{d_T(q,x)}^q \bar{F}(\xi) d\xi \right]. \quad (1.6)$$

The above analysis defines the cutoff values of inventory and demand for the three scenarios and specifies the expected ending cash position of the newsvendor for each scenario. Note that the cutoff values of demand are d_T and d_B , which respectively denote the minimum demand above which the newsvendor makes a profit and pays tax, and the minimum demand above which the newsvendor does not go bankrupt. Both cutoffs are functions of equity and inventory. We shall drop the arguments of these functions for notational convenience. For instance, d_B denotes $d_B(q, x)$. Let $\tilde{q}(x)$ denote the optimal order quantity for the newsvendor as a function of its starting capital when there is no credit limit. Solving the newsvendor's problem in the three scenarios, we show that $\tilde{q}(x)$ has the following form.

Proposition 1.3.1 *Let $q^{BWR}(x)$, $q^{BWO}(x)$, q^{NB} be order quantities defined by:*

$$q^{BWR}(x) = \bar{F}^{-1} \left(\frac{(1 + \alpha)c - s}{(1 - \tau)(p - s)} \left[\bar{F}(d_B) - \tau \bar{F}(d_T) \right] \right), \quad (1.7)$$

$$q^{BWO}(x) = \bar{F}^{-1} \left(\frac{(1 + \alpha)c - s}{(1 - \tau)(p - s)} \left[1 - \tau \bar{F}(d_T) \right] \right), \quad (1.8)$$

$$q^{NB} = \bar{F}^{-1} \left(\frac{c - s}{(1 - \tau)(p - s)} \left[1 - \tau \bar{F} \left(\frac{c - s}{p - s} q^{NB} \right) \right] \right). \quad (1.9)$$

The optimal order quantity for the newsvendor without a credit limit, $\tilde{q}(x)$, is given by $q^{BWR}(x)$ if $0 \leq x < x_2$, $q^{BWO}(x)$ if $x_2 \leq x < x_3$, x/c if $x_3 \leq x \leq x_4$, q^{NB} if $x > x_4$, where $x_2 \leq x_3 \leq x_4$ are cutoff values of the newsvendor's equity. These cutoff values are uniquely defined by $x_2 = \frac{(1+\alpha)c-s}{1+\alpha} q^{BWR}(x_2)$, $x_3 = c q^{BWO}(x_3)$, $x_4 = c q^{NB}$.

Proof: The first derivative of $E_F [\pi^{NB}]$ with respect to q is

$$(1 - \tau)(p - s) \bar{F}(q) - (c - s) \left[1 - \tau \bar{F} \left(\frac{c - s}{p - s} q \right) \right],$$

whereas the second derivative is

$$-(1-\tau)(p-s)f(q) - \tau \frac{(c-s)^2}{p-s} f\left(\frac{c-s}{p-s}q\right) \leq 0.$$

Hence, $E_F[\pi^{NB}]$ is concave. Solving the first order condition for this subscenario gives q^{NB} . Then x_4 is equal to cq^{NB} . The first derivative of $E_F[\pi^{BWO}]$ with respect to q is

$$(1-\tau)(p-s)\bar{F}(q) - [(1+\alpha)c-s]\left(1-\tau\bar{F}(d_T(q,x))\right), \quad (1.10)$$

whereas the second derivative is

$$-(1-\tau)(p-s)f(q) - \tau \frac{[(1+\alpha)c-s]^2}{p-s} f(d_T(q,x)) \leq 0.$$

Hence, $E_F[\pi^{BWO}]$ is also concave. Solving the first order condition gives $q^{BWO}(x)$. x_3 can be obtained by setting $x = cq^{BWO}(x)$, and solving the first order condition. $(1-\tau)(p-s) > (1+\alpha)c-s$ guarantees the existence of $q^{BWO}(x)$, $q^{BWR}(x)$, and q^{NB} .

We need to show that $x_3 \leq x_4$ to define the case in which the newsvendor does not borrow, but uses all of her cash for procurement (i.e., $q = x/c$). Observe that $q^{BWO}(x_3)$ is obtained by solving

$$(1-\tau)(p-s)\bar{F}(q) - [(1+\alpha)c-s]\left(1-\tau\bar{F}\left(\frac{c-s}{p-s}q\right)\right) = 0, \quad (1.11)$$

whereas $q^{NB}(x_4)$ is obtained by solving the same equation in which α is replaced with 0. Implicit differentiation of q with respect to α in (1.11) gives

$$\frac{dq}{d\alpha} = -\frac{(p-s)c\left(1-\tau\bar{F}\left(\frac{c-s}{p-s}q\right)\right)}{(1-\tau)(p-s)^2f(q) + \tau[(1+\alpha)c-s](c-s)f\left(\frac{c-s}{p-s}q\right)} < 0.$$

Hence, $q^{BWO}(x_3) < q^{NB}(x_4)$ because $\alpha > 0$, which implies that $x_3 < x_4$.

The first derivative of $E_F[\pi^{BWR}]$ with respect to q is

$$(1-\tau)(p-s)\bar{F}(q) + [(1+\alpha)c-s]\left(\tau\bar{F}(d_T) - \bar{F}(d_B)\right). \quad (1.12)$$

Then the second derivative is

$$\frac{d^2 E_F[\pi^{BWR}]}{dq^2} = -(1-\tau)(p-s)f(q) + \frac{[(1+\alpha)c-s]^2}{p-s} (f(d_B) - \tau f(d_T)). \quad (1.13)$$

$q^{BWR}(x)$ must satisfy the first order condition. Therefore, at $q = q^{BWR}(x)$

$$\begin{aligned} \frac{d^2 E_F[\pi^{BWR}]}{dq^2} &= (1-\tau)(p-s)\bar{F}(q) \left(\frac{-(1-\tau)(p-s)f(q) + \frac{[(1+\alpha)c-s]^2}{p-s} (f(d_B) - \tau f(d_T))}{(1-\tau)(p-s)\bar{F}(q)} \right) \\ &= (1-\tau)(p-s)\bar{F}(q) \left(-z(q) + \frac{(1+\alpha)c-s}{p-s} \frac{f(d_B) - \tau f(d_T)}{\bar{F}(d_B) - \tau \bar{F}(d_T)} \right) \\ &< (1-\tau)(p-s)\bar{F}(q) \left(-z(q) + \frac{(1+\alpha)c-s}{p-s} \left(\frac{f(d_B)}{\bar{F}(d_B) - \tau \bar{F}(d_T)} \right) \right) \\ &< (1-\tau)(p-s)\bar{F}(q) \left(-z(q) + \frac{(1+\alpha)c-s}{(1-\tau)(p-s)} z(d_B) \right) \\ &< 0, \end{aligned}$$

where z is the hazard rate function. $z(q) > \frac{(1+\alpha)c-s}{(1-\tau)(p-s)} z(d_B)$ because ξ is IFR and $\frac{(1+\alpha)c-s}{(1-\tau)(p-s)} < 1$. x_2 can be obtained by solving the first order condition of the borrowing with risk case after setting $q = \frac{1+\alpha}{(1+\alpha)c-s} x$.

Lastly, we need to show that $x_2 \leq x_3$. (1.8) implies that x_2 and x_3 solve

$$(1-\tau)(p-s)\bar{F}\left(\frac{1+\alpha}{(1+\alpha)c-s}x_2\right) + \tau\bar{F}\left(\frac{x_2}{p-s}\right) = (1+\alpha)c-s \quad (1.14)$$

and

$$(1-\tau)(p-s)\bar{F}\left(\frac{x_3}{c}\right) + \tau\bar{F}\left(\frac{c-s}{p-s}\left(\frac{x_3}{c}\right)\right) = (1+\alpha)c-s, \quad (1.15)$$

respectively. A pairwise comparison of the scalars multiplied by x_2 and x_3 in (1.14) and (1.15) indicates that $\frac{1+\alpha}{(1+\alpha)c-s} \geq \frac{1}{c}$ and $\frac{1}{p-s} \geq \frac{c-s}{(p-s)c}$. Hence, $x_2 \leq x_3$ as at least one of the equations would be violated otherwise. \square

Intuitively, the newsvendor borrows with bankruptcy risk if its equity is less than x_2 , borrows without bankruptcy risk if its equity lies between x_2 and x_3 , and does not borrow if its equity is greater than x_3 . In the last case, $x > x_4$ and

the optimal order quantity does not vary with x because the cash constraint is not binding.

In the next proposition, we show that the optimal order quantity is a non-monotone function of equity.

Proposition 1.3.2 *$\tilde{q}(x)$ is continuous in x . It decreases in x when x is sufficiently small, increases in x for $x_2 \leq x \leq x_4$ and is constant for $x \geq x_4$.*

Proof: The continuity of $\tilde{q}(x)$ follows by taking limits from the left and the right at the breakpoints x_2, x_3 and x_4 . For example, at x_2 , the newsvendor switches from (BWR) to (BWO). Therefore, the bankruptcy threshold d_B approaches zero as x approaches x_2 from below. Mathematically, $\lim_{x \uparrow x_2} d_B(q^{BWR}(x), x) = 0$, which implies that

$$\begin{aligned} \lim_{x \uparrow x_2} \frac{(1 + \alpha)c - s}{(1 - \tau)(p - s)} \bar{F}(q^{BWR}(x_2)) &= \lim_{x \uparrow x_2} \bar{F}(d_B) - \tau \bar{F}(d_T) \\ &= 1 - \tau \bar{F}(d_T). \end{aligned}$$

Therefore, $\lim_{x \uparrow x_2} q^{BWR}(x_2) = q^{BWO}(x_2)$. Similar analysis shows the continuity of $\tilde{q}(x)$ at the other cutoff points.

To show that $\tilde{q}(x)$ is decreasing in x for small values of x , we apply the Implicit Function Theorem to (1.7). We get

$$\frac{dq^{BWR}(x)}{dx} = \frac{[(1 + \alpha)c - s] [(1 + \alpha)f(d_B) - \tau \alpha f(d_T)]}{((1 + \alpha)c - s)^2 (f(d_B) - \tau f(d_T)) - (1 - \tau)(p - s)^2 f(q^{BWR})}. \quad (1.16)$$

The denominator in this equation is equal to $(p - s) \frac{d^2 E_F[\pi^{BWR}(x)]}{dq^2} \Big|_{q=q^{BWR}(x)}$, which is shown to be negative in the proof of Proposition 1.3.1. The numerator is positive at $x = 0$ because $d_T(q, 0) = d_B(q, 0)$. Thus, (1.16) is negative when $x = 0$. Since the function is continuously differentiable, it follows that $\tilde{q}(x)$ decreases in x when x is sufficiently small.

$\tilde{q}(x)$ is increasing in x for $x \in [x_2, x_3]$ because $E_F[\pi^{BWO}]$ is supermodular, i.e.,

$$\frac{\partial^2 E_F[\pi^{BWO}]}{\partial q \partial x} = \alpha \tau \frac{(1 + \alpha)c - s}{(p - s)^2} f(d_T) \geq 0. \quad (1.17)$$

For $x \in [x_3, x_4]$, we have $\tilde{q}(x) = x/c$, which implies that $\tilde{q}(x)$ increases in $x \in [x_3, x_4]$. Lastly, for $x \geq x_4$, $\tilde{q}(x) = q^{NB}$, which does not vary with x . \square

The main inference from this proposition is that the optimal order quantity is decreasing in equity for small values of x . This property will be useful in solving for the equilibrium order quantity. Intuitively, it means that the investor's tendency to take riskier operational decisions increases when she injects less equity. This occurs due to the existence of a moral hazard problem under limited liability. See Easterbrook and Fischel [32] for a general discussion on the incentives created by limited liability to transfer risk to debt holders.

1.3.2 The Bank's Problem

The bank's objective is to maximize its expected profit by lending to the newsvendor. Let $\kappa(q, w, x, \xi)$ denote the bank's profit as a function of the inventory level q , the loan amount $w = (cq - x)^+$, the equity of the firm x and the realized demand ξ . There are two scenarios. If the realized demand exceeds d_B , then the firm survives and repays the loan plus interest to the bank. As a result, the inventory collateral is not used and the bank's profit is equal to $\alpha(cq - x)^+$. In contrast, if the realized demand is below d_B , then the firm declares bankruptcy and its cash $p\xi$ and unsold inventory $q - \xi$ are possessed by the bank. The bank liquidates $q - \xi$ units to recover its losses. We assume that the salvage value of leftover inventory for the bank is the same as for the firm because both can access the same market. However, due to forced liquidation, the bank bears an

additional bankruptcy cost proportional to the size of the bankruptcy. Hence, if $\xi < d_B$, then the bank's profit is

$$\kappa(q, w, x, \xi) = p\xi + s(q - \xi) - b(d_B - \xi) - w. \quad (1.18)$$

Here, $b \geq 0$ is the bankruptcy cost per unit and the size of the bankruptcy is given by the difference between the bankruptcy threshold demand d_B and the realized demand ξ .

We rewrite (1.18) by using the definition of d_B and by noting that $w = cq - x$ when $d_B > 0$. Thus, we find that the bank's profit is given by:

$$\kappa(q, (cq - x)^+, x, \xi) = \alpha(cq - x)^+ - (p - s + b)[d_B - \xi]^+,$$

which is equal to zero if the firm does not borrow, equal to $\alpha(cq - x)$ if the firm borrows and survives, and equal to (1.18) if the firm borrows and defaults. Observe that the profit decomposes into two terms, the first gives the profit when there is no bankruptcy, and the second captures the amount of loan write-off in case the newsvendor defaults on its loan. The bank's problem is to determine the credit limit for the newsvendor in order to maximize the expectation of (1.19) under its belief G . This problem is equivalent to finding the order quantity that maximizes the bank's expected profit. The following proposition solves the bank's problem.

Proposition 1.3.3 *The bank's expected profit, $E_G[\kappa(q, (cq - x)^+, x, \xi)]$, is maximized when $q = q^L(x) \equiv \beta x + \theta$, where $\beta = \frac{1+\alpha}{(1+\alpha)c-s}$ and $\theta = \frac{p-s}{(1+\alpha)c-s} \bar{G}^{-1}\left(1 - \frac{\alpha c}{(1+\alpha)c-s} \left(\frac{p-s}{p-s+b}\right)\right)$. The corresponding asset based credit limit offered by the bank is equal to*

$$\psi(x) = \frac{s}{(1+\alpha)c-s}x + c\theta. \quad (1.19)$$

Proof: Taking a derivative with respect to q and setting it equal to zero gives $q^L(x)$ as one can show that the bank's expected return under the newsvendor bankruptcy is concave in q for a given x . Hence, the bank does not want to lend more than $\psi(x) = cq^L(x) - x$. \square

Proposition 1.3.3 provides several insights into the effectiveness of this lending mechanism. First, observe that the credit limit has two components, $sx/[(1 + \alpha)c - s]$ and $c\theta$. The first component is riskless because it is the maximum amount that the firm can borrow and repay with interest in full with probability 1, i.e., even when the realized demand is zero. The second component $c\theta$ is risky because, if the firm borrows more than the first component, its loan repayment depends on the realized demand. Note that only the riskless component increases with equity, whereas the risky component is a constant. Thus, the risky component can be interpreted as the minimum credit line offered to an all-debt newsvendor. That is, the bank is willing to lend money to the newsvendor based on its demand prospects even if its equity is zero. This inference is consistent with practice, as banks commonly lend money to firms with zero or negative equity. Note that θ decreases in b , which implies that the bank will offer a smaller credit line when the bankruptcy cost is high.

Second, only the risky component of the credit limit depends on information asymmetry. This outcome is intuitive because the bank can observe the newsvendor's equity and inventory. Information asymmetry comes into the play only when the bank evaluates the newsvendor's demand prospects. Lenders' conservative estimates about potential investments during economic downturns correspond to a lower θ value in our model. This is consistent with the shrinkage of the aggregate loan supply during economic downturns. (See

Leary [52] and references therein for the relationship between economic downturns and loan supply.)

Third, the bank has full control over its maximum loan write-off because it is independent of the newsvendor's starting equity x or the true demand distribution F . To see this, note that the bank's maximum loan write-off occurs when the newsvendor borrows up to the credit limit and realizes a demand equal to zero. Thus, it is equal to $(p - s + b)d_B(\beta x + \theta, x) = (p - s + b)\bar{G}^{-1}\left[1 - \frac{\alpha c}{(1+\alpha)c-s}\left(\frac{p-s}{p-s+b}\right)\right]$. This quantity does not change with x or F . Therefore, even as the amount of the loan may vary with the equity of the firm or its true demand distribution, the asset based credit limit creates an upper bound on the bank's loan write-off.

Finally, we find that if the firm uses its entire credit limit then the distribution of the bank's loan write-off is also independent of x . This result is shown in the following lemma.

Lemma 1.3.1 *If the firm uses its entire credit limit, then it orders $q = \beta x + \theta$ and the probability distribution of the bank's loan write-off is independent of x . In other words, let $\mathcal{W} \equiv (p - s + b)[d_B - \xi]^+$ denote the bank's loan write-off in case the firm defaults on its loan, and let the maximum of \mathcal{W} be $\mathcal{W}_{max} \equiv (p - s + b)\bar{G}^{-1}\left[1 - \frac{\alpha c}{(1+\alpha)c-s}\left(\frac{p-s}{p-s+b}\right)\right]$. Then*

$$Pr(\mathcal{W} \leq \zeta) = \begin{cases} \bar{F}\left(\bar{G}^{-1}\left[1 - \frac{\alpha c}{(1+\alpha)c-s}\left(\frac{p-s}{p-s+b}\right)\right] - \frac{\zeta}{p-s+b}\right) & \text{if } \zeta \leq \mathcal{W}_{max}, \\ 1 & \text{if } \zeta > \mathcal{W}_{max}, \end{cases} \quad (1.20)$$

which is independent of x .

Proof: If the firm uses its entire credit line, then $d_B = \frac{(1+\alpha)c-s}{p-s}(\beta x + \theta) - \frac{1+\alpha}{p-s}x = \bar{G}^{-1}\left[1 - \frac{\alpha c}{(1+\alpha)c-s}\left(\frac{p-s}{p-s+b}\right)\right]$. The bank's maximum loan write-off occurs when the

realized demand is zero. Setting $\xi = 0$ in $\mathcal{W} = (p - s + b)[d_B - \xi]^+$ gives \mathcal{W}_{max} . For $0 \leq \zeta \leq \mathcal{W}_{max}$,

$$\begin{aligned}
Pr\{\mathcal{W} \leq \zeta\} &= Pr\{(p - s + b)[d_B - \xi]^+ \leq \zeta\} \\
&= Pr\left\{(p - s + b)\left(\bar{G}^{-1}\left[1 - \frac{\alpha c}{(1 + \alpha)c - s}\left(\frac{p - s}{p - s + b}\right)\right] - \xi\right) \leq \zeta\right\} \\
&= \bar{F}\left(\bar{G}^{-1}\left[1 - \frac{\alpha c}{(1 + \alpha)c - s}\left(\frac{p - s}{p - s + b}\right)\right] - \frac{\zeta}{p - s + b}\right).
\end{aligned}$$

□

Intuitively, this lemma means that if the firm seeks a larger borrowing, the bank demands the owner to increase her equity investment proportionately so that the credit limit increases but the write-off distribution remains unchanged. As a consequence, contrary to unsecured loans, the riskiness of an asset based loan does not increase in the borrowing amount, which implies that the bank does not need to increase the interest rate to compensate for risk as the amount of borrowing increases. We shall show in the next section that, at equilibrium, the credit limit is always binding when the firm borrows with bankruptcy risk. Therefore, the result of Lemma 1.3.1 applies. These insights provide theoretical justification for the practicality of ABL under information asymmetry.

1.3.3 The Newsvendor-Bank Interaction

Following Propositions 1.3.1 and 1.3.3, the order quantity at equilibrium is given by $\tilde{q}(x)$ if the credit limit is not binding, and by $q^L(x)$ otherwise. Lemma 1.3.2 proves a necessary and sufficient condition for these two functions to intersect, and show when the credit limit is and is not binding.

Lemma 1.3.2 *If $q^{BWR}(0) \leq q^L(0)$ then the credit limit is never binding. If $q^{BWR}(0) > q^L(0)$, then there exists an equity value x_1 such that $0 < x_1 < x_2$ and $q^{BWR}(x_1) = q^L(x_1)$ and the credit limit is binding for $x \in [0, x_1]$. The value of x_1 is given by*

$$\frac{(1-\tau)(p-s)}{(1+\alpha)c-s} \bar{F}(\beta x_1 + \theta) + \tau \bar{F}\left(\frac{x_1 + [(1+\alpha)c-s]\theta}{p-s}\right) = \bar{F}\left(\frac{(1+\alpha)c-s}{p-s}\theta\right). \quad (1.21)$$

Proof: $q^{BWR}(x)$ and $q^L(x)$ are both continuous functions. First, we show that $q^{BWR}(x_2) < q^L(x_2)$. To see this, note that $x_2 = q^{BWR}(x_2)/\beta$, whereas $q^L(x_2) = \beta x_2 + \theta = q^{BWR}(x_2) + \theta > q^{BWR}(x_2)$. Therefore, it follows that if $q^{BWR}(0) > q^L(0)$, then there exists $0 < x_1 < x_2$ such that $q^{BWR}(x_1) = q^L(x_1)$. Substituting $q^{BWR}(x_1) = q^L(x_1) = \beta x_1 + \theta$ in (1.7) gives (1.21). x_1 is unique because the left hand side of (1.21) increases in x . For the other direction, suppose $q^L(0) \leq q^{BWR}(0)$, and the two functions intersect. The intersection point must be unique because, as we explained above, it has to satisfy (1.21). If the intersection point is unique, then the derivatives of the two functions at that point must be equal to each other (i.e., the derivative of $q^L(x) - q^{BWR}(x)$ must be zero). From (1.16) and Proposition 1.3.3, $\frac{dq^L(x)}{dx} = \frac{dq^{BWR}(x)}{dx}$ is equivalent to

$$\beta = \frac{[(1+\alpha)c-s][(1+\alpha)f(d_B) - \tau\alpha f(d_T)]}{((1+\alpha)c-s)^2(f(d_B) - \tau f(d_T)) - (1-\tau)(p-s)^2 f(q^{BWR})},$$

which can be written as

$$\tau((1+\alpha)c-s)^2 f(d_T) + (1-\tau)(1+\alpha)(p-s)^2 f(q^{BWR}) = 0,$$

which cannot hold because ξ has an IFR distribution. Therefore, the two functions cannot intersect if $q^L(0) \leq q^{BWR}(0)$. \square

Figure 1.2 illustrates the outcome of the newsvendor-bank sub-game. In Figure 1.2(a), the amount that the bank is willing to lend to the firm with zero equity is less than the amount that the firm wants to borrow with zero equity.

Thus, x_1 exists. The newsvendor's order quantity is restricted by the credit limit when $x \leq x_1$ and unrestricted otherwise. In Figure 1.2(b), $q^{BWR}(0) \leq q^L(0)$, so that the newsvendor's order quantity is nowhere restricted by the credit limit. The fact that $q^{BWR}(0)$ can be less than $q^L(0)$ is interesting because it shows that the credit line offered to an all-debt newsvendor can exceed its unconstrained optimal borrowing amount.

We can now specify the equilibrium solution. When $q^{BWR}(0) > q^L(0)$, the newsvendor's order quantity is given by:

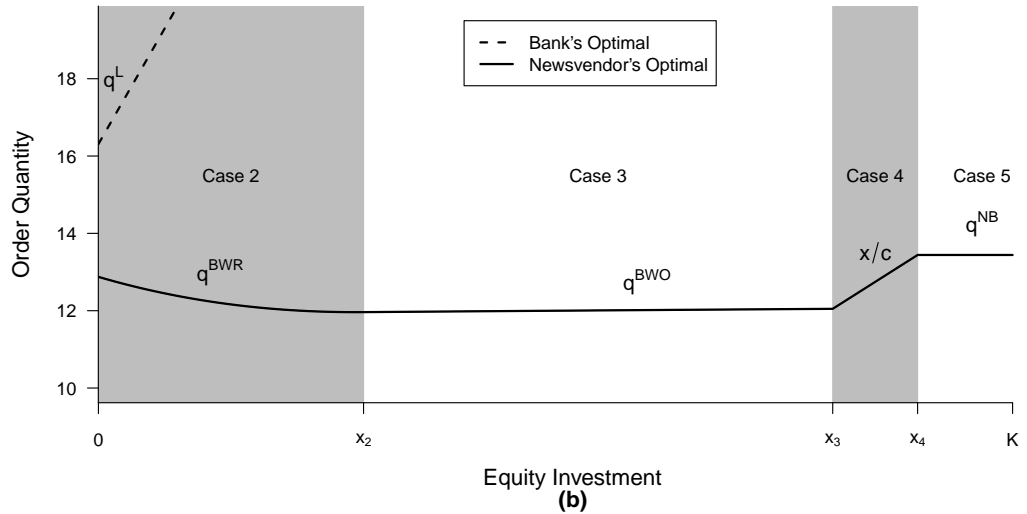
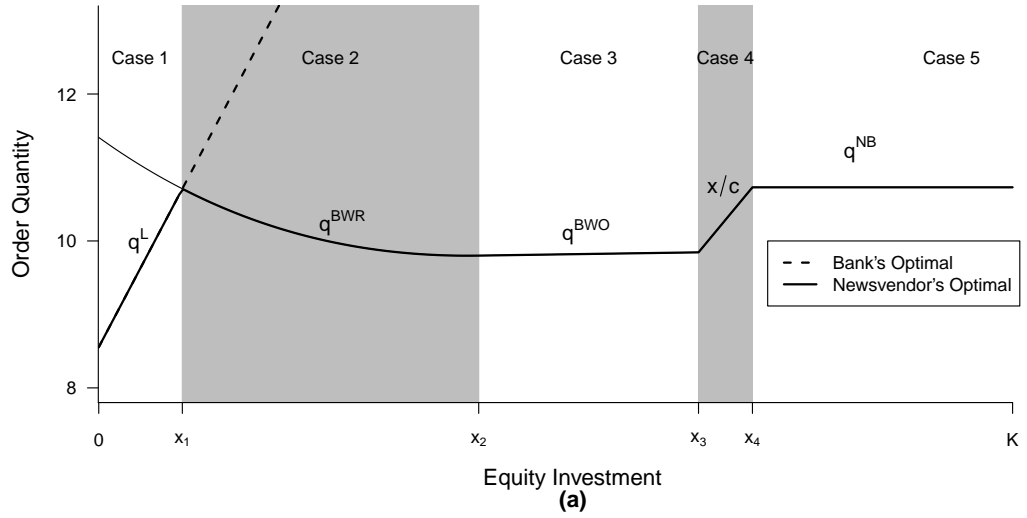
$$q^*(x) = \begin{cases} q^L(x) & \text{if } 0 \leq x \leq x_1 \text{ (Case 1),} \\ q^{BWR}(x) & \text{if } x_1 < x < x_2 \text{ (Case 2),} \\ q^{BWO}(x) & \text{if } x_2 \leq x < x_3 \text{ (Case 3),} \\ x/c & \text{if } x_3 \leq x \leq x_4 \text{ (Case 4),} \\ q^{NB} & \text{if } x > x_4 \text{ (Case 5).} \end{cases} \quad (1.22)$$

Correspondingly, the newsvendor's expected ending cash position is

$$\pi^*(x) = \begin{cases} (p-s) \left[\int_{d_B}^{q^L} \bar{F}(\xi) d\xi - \tau \int_{d_T}^{q^L} \bar{F}(\xi) d\xi \right] & \text{if } 0 \leq x \leq x_1, \\ (p-s) \left[\int_{d_B}^{q^{BWR}} \bar{F}(\xi) d\xi - \tau \int_{d_T}^{q^{BWR}} \bar{F}(\xi) d\xi \right] & \text{if } x_1 < x < x_2, \\ (1+\alpha)x - ((1+\alpha)c-s)q^{BWO} + (p-s) \left[\int_0^{q^{BWO}} \bar{F}(\xi) d\xi - \tau \int_{d_T}^{q^{BWO}} \bar{F}(\xi) d\xi \right] & \text{if } x_2 \leq x < x_3, \\ \frac{s}{c}x + (p-s) \left[\int_0^{x/c} \bar{F}(\xi) d\xi - \tau \int_{\frac{(c-s)x}{(p-s)c}}^{x/c} \bar{F}(\xi) d\xi \right] & \text{if } x_3 \leq x \leq x_4, \\ x - (c-s)q^{NB} + (p-s) \left[\int_0^{q^{NB}} \bar{F}(\xi) d\xi - \tau \int_{\frac{c-s}{p-s}q^{NB}}^{q^{NB}} \bar{F}(\xi) d\xi \right] & \text{if } x > x_4. \end{cases} \quad (1.23)$$

Note that $q^*(x)$ and $\pi^*(x)$ are defined in terms of five cases. In Case 1, the newsvendor borrows with bankruptcy risk and the bank's credit limit is bind-

Figure 1.2: The newsvendor's and the bank's optimal order quantities for different investment levels. In Figure (a), demand is Weibull with $\bar{F}(\xi) = \exp(-(\xi/\lambda)^k)$. $E_F[\xi] = 10$ and shape parameter $k = 2$. $p = 2$, $c = 1$, $s = 0.4$, $b = 0.2$, $\tau = 0.4$, and $\alpha = 0.10$. The bank believes that demand is Weibull with $E_G[\xi] = 9$ and shape parameter $k = 2$. As a result, $q^{BWR}(0) = 11.41 > q^L(0) = 8.55$, and the credit limit binds for $x \in [0, x_1]$. $x_1 = 1.37$, $x_2 = 6.24$, $x_3 = 9.85$, $x_4 = 10.73$, and $q^{NB} = 11.17$. In Figure (b), all the model parameters are the same as Figure (a) except $s = 0.7$. As a result, $q^L(0) = 16.31 > q^{BWR}(0) = 12.88$, and the credit limit never binds. $x_2 = 4.35$, $x_3 = 12.05$, $x_4 = 12.71$, and $q^{NB} = 12.93$.



ing. In Case 2, the newsvendor borrows with risk, but the bank's credit limit is not binding. Cases 1 and 2 are subsumed in scenario (BWR) defined earlier. In Case 3, the newsvendor borrows without bankruptcy risk; this corresponds to scenario (BWO). In Case 4, the newsvendor does not borrow and uses up all the equity to procure inventory. In Case 5, the newsvendor does not borrow, is left with excess cash. Cases 4 and 5 correspond to scenario (NB). When $q^{BWR}(0) \leq q^L(0)$, the equilibrium solution is similar except that Case 1 does not arise.

In the next section, we will see that some of the five cases do not occur at equilibrium once the owner's equity investment problem is introduced. We present the solution to the investor's problem, first under the condition $q^{BWR}(0) > q^L(0)$, then under $q^{BWR}(0) \leq q^L(0)$.

1.3.4 The Owner's Equity Investment Decision

The owner, who has capital K to invest, solves a capital allocation problem to determine the amount x of equity of the newsvendor and $K - x$ of investment in an external market asset. We denote the rate of return on the external asset by a random variable α_m with mean $\bar{\alpha}_m \geq 0$. The owner's expected value maximization problem is formulated as:

$$\Pi^* \equiv \max_{x \in [0, K]} \Pi(x) = \max_{x \in [0, K]} \pi^*(x) + E[(1 + \alpha_m)(K - x)]. \quad (1.24)$$

The first part of the objective function, $\pi^*(x)$, denotes the payoff from investing in the newsvendor and is given by (1.23), whereas the second part denotes the payoff from investing in the external market. We assume that K is sufficiently large to procure the optimal order quantity of a pure equity newsvendor, q^{NB} .

This assumption is made only to ease the presentation because it guarantees the feasibility of Case 5 in (1.22).

We solve the owner's problem by determining the optimal solution in each of the Cases 1-5 and then finding the highest value. We first show that Case 2 (i.e., $x \in (x_1, x_2)$) cannot arise in equilibrium.

Lemma 1.3.3 $\Pi(x)$ is convex in x for $x \in (x_1, x_2)$. Thus, borrowing with risk but ordering less than the bank's optimal order quantity q^L cannot arise in equilibrium.

Proof: The order quantity in Case 2, $q^{BWR}(x)$, solves

$$(1 - \tau)(p - s)\bar{F}(q^{BWR}(x)) + [(1 + \alpha)c - s](\tau\bar{F}(d_T) - \bar{F}(d_B)) = 0, \quad (1.25)$$

and the owner solves

$$\begin{aligned} \Pi_2^* &= \max_{x \in [x_1, x_2]} \Pi_2(x) \\ &= \max_{x \in [x_1, x_2]} (1 + \bar{\alpha}_m)(K - x) + (p - s) \left(\int_{d_B}^{q^{BWR}(x)} \bar{F}(\xi) d\xi - \tau \int_{d_T}^{q^{BWR}(x)} \bar{F}(\xi) d\xi \right) \end{aligned}$$

The first derivative of $\Pi_2(x)$ with respect to x is

$$\begin{aligned} \frac{d\Pi_2(x)}{dx} &= -(1 + \bar{\alpha}_m) + (1 + \alpha)\bar{F}(d_B) - \tau\alpha\bar{F}(d_T) \\ &= -(1 + \bar{\alpha}_m) + (1 + \alpha) \frac{(1 - \tau)(p - s)}{(1 + \alpha)c - s} \bar{F}(q^{BWR}(x)) + \tau\bar{F}(d_T). \end{aligned}$$

The second line follows from (1.25). We drop the superscript BWR for notational convenience. The second derivative is

$$-\left((1 - \tau)(1 + \alpha)f(q) + \tau \frac{(1 + \alpha)c - s}{p - s} f(d_T) \right) \frac{dq}{dx} + \tau \frac{\alpha}{p - s} f(d_T) \quad (1.26)$$

Using (1.16), (1.26) can be written as

$$\begin{aligned} \frac{d\Pi_2^2(x)}{dx^2} &= (1 - \tau) \frac{\tau\alpha((p-s)-(1+\alpha)[(1+\alpha)c-s])f(d_T)f(q) + (1+\alpha)^2[(1+\alpha)c-s]f(q)f(d_B)}{(1-\tau)(p-s)^2f(q) - [(1+\alpha)c-s]^2(f(d_B) - \tau f(d_T))} \\ &\quad + \frac{\tau[(1+\alpha)c-s]^2 f(d_T)f(d_B)}{(p-s)[(1-\tau)(p-s)^2f(q) - [(1+\alpha)c-s]^2(f(d_B) - \tau f(d_T))]}, \end{aligned}$$

which is non-negative because we assume that $\frac{p-s}{(1+\alpha)c-s} \geq 1 + \alpha$ and the denominators of both terms are positive from (1.16). Hence, we can rule out the interior, (x_1, x_2) , as the optimal solution will be at $x = x_1$ or $x = x_2$. Furthermore, the objective function is continuous, which implies that $x = x_1$ is taken in the account in the first case. \square

As a consequence of this lemma, if the firm borrows with bankruptcy risk then the credit limit is binding. Thus, the optimal equity investment with bankruptcy risk lies in the range $x \in [0, x_1]$. Its value is given by the following proposition.

Proposition 1.3.4 *Let $\alpha_l = \alpha \frac{(1-\tau)(p-s)}{(1+\alpha)c-s} \bar{F}(\beta x_1 + \theta) + \bar{F}\left(\frac{(1+\alpha)c-s}{p-s}\theta\right) - 1$ and $\alpha_h = (1 + \alpha) \frac{(1-\tau)(p-s)}{(1+\alpha)c-s} \bar{F}(\theta) + \tau \bar{F}\left(\frac{(1+\alpha)c-s}{p-s}\theta\right) - 1$. The optimal equity investment with bankruptcy risk, x_R^* , is given by*

$$x_R^* = \begin{cases} x_1 & \text{if } \bar{\alpha}_m < \alpha_l, \\ \tilde{x}_R & \text{if } \bar{\alpha}_m \in [\alpha_l, \alpha_h], \\ 0 & \text{if } \bar{\alpha}_m > \alpha_h, \end{cases} \quad (1.27)$$

where \tilde{x}_R solves

$$(1 - \tau) \frac{(1 + \alpha)(p - s)}{(1 + \alpha)c - s} \bar{F}(\beta \tilde{x}_R + \theta) + \tau \bar{F}\left(\frac{\tilde{x}_R + [(1 + \alpha)c - s]\theta}{p - s}\right) - 1 = \bar{\alpha}_m. \quad (1.28)$$

Proof: This is Case 1 in which the owner solves

$$\begin{aligned} \Pi_1^* &= \max_{x \in [0, x_1]} \Pi_1(x) \\ &= \max_{x \in [0, x_1]} (1 + \bar{\alpha}_m)(K - x) + (p - s) \left[\int_{\frac{(1+\alpha)c-s}{p-s}\theta}^{q^L(x)} \bar{F}(\xi) d\xi - \tau \int_{d_T(x)}^{q^L(x)} \bar{F}(\xi) d\xi \right], \end{aligned}$$

where $q^L(x) = \beta x + \theta$ and $d_T(x) = \frac{x + [(1+\alpha)c-s]\theta}{p-s}$. The objective function is concave in $[0, x_1]$ because the second derivative is

$$-(1 - \tau) \frac{(1 + \alpha)(p - s)}{(1 + \alpha)c - s} \beta f(\beta x + \theta) - \frac{\tau}{p - s} f\left(\frac{x_1 + [(1 + \alpha)c - s]\theta}{p - s}\right) \leq 0.$$

Setting the first derivative equal to zero gives (1.28). α_h is obtained by setting $x = 0$ in (1.28). Similarly, α_l is obtained by setting $x = x_1$. That is,

$$\begin{aligned}\alpha_l &= (1 - \tau) \frac{(1 + \alpha)(p - s)}{(1 + \alpha)c - s} \bar{F}(\beta x_1 + \theta) + \tau \bar{F}\left(\frac{x_1 + [(1 + \alpha)c - s]\theta}{p - s}\right) - 1 \\ &= (1 - \tau) \frac{\alpha(p - s)}{(1 + \alpha)c - s} \bar{F}(\beta x_1 + \theta) + \bar{F}\left(\frac{(1 + \alpha)c - s}{p - s} \theta\right) - 1.\end{aligned}$$

The second equality follows from (1.21). It can be shown that $\tilde{x}_R < 0$ when $\bar{\alpha}_m > \alpha_h$ and $\tilde{x}_R > x_1$ when $\bar{\alpha}_m < \alpha_l$. Moreover, $\alpha_h > \alpha_l$ because the left hand side of (1.28) is decreasing in x . This proves the result. \square

Intuitively, when $\bar{\alpha}_m$ is relatively high (i.e., $\bar{\alpha}_m > \alpha_h$), the owner does not invest any amount in the newsvendor because her opportunity cost is high. When $\bar{\alpha}_m$ is relatively low (i.e., $\bar{\alpha}_m < \alpha_l$), the owner invests as much as she can (i.e., $x = x_1$). For intermediate values of $\bar{\alpha}_m$, the owner chooses a value of equity in order to match the return from the newsvendor, given by the left hand side of (1.28), with $\bar{\alpha}_m$.

Now consider the cases when the newsvendor does not face any risk of bankruptcy, i.e., when $x \in [x_2, K]$. The optimal equity investment value without bankruptcy risk, x_{NR}^* is given by the following proposition.

Proposition 1.3.5 *Let $\alpha_2 = \alpha \left[1 - \tau \bar{F}\left(\frac{x_2}{p-s}\right)\right]$ and $\alpha_3 = \alpha \left[1 - \tau \bar{F}\left(\frac{c-s}{p-s} q^{BWO}(x_3)\right)\right]$. The optimal equity investment in $[x_2, K]$, i.e., without bankruptcy risk, is*

$$x_{NR}^* = \begin{cases} x_2 & \text{if } \bar{\alpha}_m > \alpha_2, \\ x_3^* & \text{if } \bar{\alpha}_m \in (\alpha_3, \alpha_2], \\ x_4^* & \text{if } \bar{\alpha}_m \in [0, \alpha_3]. \end{cases} \quad (1.29)$$

where x_3^* solves

$$\bar{F}\left(q^{BWO}(x_3^*)\right) = \frac{\bar{\alpha}_m[(1 + \alpha)c - s]}{\alpha(1 - \tau)(p - s)} \quad (\text{Case 3}),$$

and x_4^* solves

$$(1 - \tau)(p - s)\bar{F}\left(\frac{x_4^*}{c}\right) + \tau(c - s)\bar{F}\left(\frac{c - s}{p - s}\left[\frac{x_4^*}{c}\right]\right) = (1 + \bar{\alpha}_m)c - s \quad (\text{Case 4}).$$

Proof: In Case j , $j \in \{3, 4, 5\}$, let $\Pi_j(x)$ denote the owner's payoff function as a function of the equity amount, and Π_j^* denote the owner's optimal payoff. In Case 3, the owner solves

$$\begin{aligned} \Pi_3^* &= \max_{x \in [x_2, x_3]} \Pi_3(x) \\ &= \max_{x \in [x_2, x_3]} (1 + \bar{\alpha}_m)(K - x) + (1 + \alpha)x + ((1 + \alpha)c + s)q^{BWO}(x) \\ &\quad + (p - s) \left[\int_0^{q^{BWO}(x)} \bar{F}(\xi) d\xi - \tau \int_{d_T}^{q^{BWO}(x)} \bar{F}(\xi) d\xi \right]. \end{aligned}$$

Taking a derivative with respect to x and using the definition of $q^{BWO}(x)$ from (1.8), we get

$$\frac{d\Pi_3}{dx} = \alpha - \bar{\alpha}_m - \alpha\tau\bar{F}(d_T(x)) = \alpha \frac{(1 - \tau)(p - s)}{(1 + \alpha)c - s} \bar{F}(q^{BWO}(x)) - \bar{\alpha}_m. \quad (1.30)$$

The second derivative with respect to x is $-\alpha \frac{(1 - \tau)(p - s)}{(1 + \alpha)c - s} \frac{dq^{BWO}(x)}{dx} f(q^{BWO}(x))$, which is negative because q^{BWO} is increasing in x by Proposition 1.3.2. Therefore, the owner's problem is concave in Case 3. Concavity implies that (1.30) attains its maximum at $x = x_2$ and its minimum at $x = x_3$, which implies that the optimal solution is in (x_2, x_3) only if 0 is between the values of the first derivative at $x = x_2$ and $x = x_3$. The first derivatives at $x = x_2$ and $x = x_3$ are $\alpha \frac{(1 - \tau)(p - s)}{(1 + \alpha)c - s} \bar{F}(q^{BWO}(x_2)) - \bar{\alpha}_m$ and $\alpha \frac{(1 - \tau)(p - s)}{(1 + \alpha)c - s} \bar{F}(q^{BWO}(x_3)) - \bar{\alpha}_m$, respectively. Let $\alpha_2 = \alpha \frac{(1 - \tau)(p - s)}{(1 + \alpha)c - s} \bar{F}(q^{BWO}(x_2)) = \alpha [1 - \tau \bar{F}(d_T(q^{BWO}(x_2), x_2))]$. The last equality follows from (1.8). Similarly, let $\alpha_3 = \alpha \frac{(1 - \tau)(p - s)}{(1 + \alpha)c - s} \bar{F}(q^{BWO}(x_3)) = \alpha [1 - \tau \bar{F}(d_T(q^{BWO}(x_3), x_3))]$. Then the optimal equity investment in Case 3, x_3^* , is given by

$$x_3^* = \begin{cases} x_2 & \text{if } \bar{\alpha}_m > \alpha_2, \\ \tilde{x}_3 & \text{if } \bar{\alpha}_m \in [\alpha_3, \alpha_2], \\ x_3 & \text{if } \bar{\alpha}_m < \alpha_3, \end{cases} \quad (1.31)$$

where \tilde{x}_3 is obtained by setting (1.30) equal to zero.

In Case 4, the owner solves

$$\begin{aligned}\Pi_4^* &= \max_{x \in [x_3, x_4]} \Pi_4(x) \\ &= \max_{x \in [x_3, x_4]} (1 + \bar{\alpha}_m)(K - x) + \frac{s}{c}x + (p - s) \left[\int_0^{x/c} \bar{F}(\xi) d\xi - \tau \int_{\frac{(c-s)x}{(p-s)c}}^{x/c} \bar{F}(\xi) d\xi \right].\end{aligned}$$

This objective function is concave in x because the second derivative is $-(1 - \tau)\frac{p-s}{c^2}f(x/c) - \tau\left(\frac{c-s}{p-s}\right)^2\frac{1}{p-s}f\left(\frac{(c-s)x}{(p-s)c}\right) \leq 0$. The first derivative is

$$-(1 + \bar{\alpha}_m) + \frac{s}{c} + (1 - \tau)\frac{p-s}{c}\bar{F}\left(\frac{x}{c}\right) + \tau\frac{c-s}{c}\bar{F}\left(\frac{(c-s)x}{(p-s)c}\right), \quad (1.32)$$

which is decreasing in x . $\tilde{q}(x)$ is continuous by Proposition 1.3.2. Therefore, $x_3/c = q^{BWO}(x_3)$, and the first derivative at $x = x_3$ is

$$-(1 + \bar{\alpha}_m) + \frac{s}{c} + \frac{p-s}{c} \left((1 - \tau)\bar{F}(q^{BWO}(x_3)) + \tau\frac{c-s}{p-s}\bar{F}\left(\frac{c-s}{p-s}q^{BWO}(x_3)\right) \right). \quad (1.33)$$

Using (1.8), we know that $\tau\bar{F}\left(\frac{c-s}{p-s}q^{BWO}(x_3)\right) = 1 - \frac{(1-\tau)(p-s)}{(1+\alpha)c-s}\bar{F}(q^{BWO}(x_3))$. Substituting it into (1.33) gives $\alpha\frac{(1-\tau)(p-s)}{(1+\alpha)c-s}\bar{F}(q^{BWO}(x_3)) - \bar{\alpha}_m$, which is equal to $\alpha_3 - \bar{\alpha}_m$. Similar analysis shows that the derivative at $x = x_4$ is equal to $-\bar{\alpha}_m$. Therefore,

$$x_4^* = \begin{cases} x_3 & \text{if } \bar{\alpha}_m > \alpha_3, \\ \tilde{x}_4 & \text{if } \bar{\alpha}_m \in [0, \alpha_3], \end{cases}$$

where \tilde{x}_4 is obtained by setting (1.32) equal to zero.

In Case 5, the owner solves

$$\begin{aligned}\Pi_5^* &= \max_{x \in [x_4, K]} \Pi_5(x) \\ &= \max_{x \in [x_4, K]} (1 + \bar{\alpha}_m)(K - x) + x - (c - s)q^{NB} \\ &\quad + (p - s) \left[\int_0^{q^{NB}} \bar{F}(\xi) d\xi - \tau \int_{\frac{c-s}{p-s}q^{NB}}^{q^{NB}} \bar{F}(\xi) d\xi \right],\end{aligned}$$

which is linear in x . Therefore, $x_5^* = x_4$ because $\bar{\alpha}_m \geq 0$.

The objective function $\Pi(x)$ is concave in all three cases, and the first derivatives from the right and left are equal to each other at every switching point. Therefore, collecting these three cases together gives the optimal solution under no borrowing. \square

Here x_3^* corresponds to the optimal equity investment value if the firm borrows without risk (i.e., if the optimal equity investment is in $(x_2, x_3]$), and x_4^* corresponds to the optimal equity investment value if the firm does not borrow (i.e., if the optimal equity investment is in $(x_3, x_4]$). To derive the solution in Proposition 1.3.5, we show that the owner's objective function is concave in Cases 3-5, and the first derivatives from the right and left are equal to each other at every switching point. Therefore, there is a unique local optimum, and collecting the three cases together gives the optimal solution under no borrowing. The owner's solution lies at the left boundary x_2 when $\bar{\alpha}_m > \alpha_2$ because the external asset offers a very attractive investment alternative. As the external investment alternative becomes less promising (i.e., when $\bar{\alpha}_m \in (\alpha_3, \alpha_2]$), the owner uses a mix of riskless debt and equity to finance her firm's operations. For smaller $\bar{\alpha}_m$ values (i.e., for $\bar{\alpha}_m \in [0, \alpha_3]$), the owner creates a pure equity firm by investing x_4^* , which sets the after-tax return from the newsvendor equal to $\bar{\alpha}_m$.

Let $\Pi_R^*(\bar{\alpha}_m)$ and $\Pi_{NR}^*(\bar{\alpha}_m)$ denote the owner's payoff functions with and without bankruptcy risk, respectively. We find that Π_R^* and Π_{NR}^* are both increasing in $\bar{\alpha}_m$, but Π_R^* is increasing at a faster rate. Moreover, $\Pi_R^* > \Pi_{NR}^*$ when $\bar{\alpha}_m$ is sufficiently large. Therefore, if $\Pi_R^*(0) > \Pi_{NR}^*(0)$ at $\bar{\alpha}_m = 0$, then the two functions never intersect and investing with risk is optimal for all $\bar{\alpha}_m \geq 0$. This scenario can arise if the bank offers a sufficiently attractive borrowing opportunity with a high

credit limit and/or a low interest rate. On the other hand, if $\Pi_R^*(0) \leq \Pi_{NR}^*(0)$, then the two functions intersect at a unique threshold value of $\bar{\alpha}_m$. Let $\tilde{\alpha}$ denote this threshold. If $\bar{\alpha}_m \leq \tilde{\alpha}$, then the owner invests enough equity into the newsvendor that the probability of bankruptcy is zero. Otherwise, the owner finds it optimal to invest with bankruptcy risk. Proposition 1.3.6 formalizes this result.

Proposition 1.3.6 *If $\Pi_R^*(0) > \Pi_{NR}^*(0)$ then investing with bankruptcy risk is optimal for all $\bar{\alpha}_m \geq 0$. Otherwise, there exists a unique threshold return value $\tilde{\alpha} \geq 0$ such that investing without bankruptcy risk is optimal when $\bar{\alpha}_m \leq \tilde{\alpha}$ and investing with bankruptcy risk is optimal otherwise.*

Proof: We first show that both Π_R^* and Π_{NR}^* increase in $\bar{\alpha}_m$, but Π_R^* increases at a faster rate. For $s \in \{R, NR\}$,

$$\frac{d\Pi_s^*}{d\bar{\alpha}_m} = \frac{\partial \Pi_s(\bar{\alpha}_m, x^*)}{\partial \bar{\alpha}_m} + \frac{\partial \Pi_s(\bar{\alpha}_m, x)}{\partial x} \Big|_{x=x^*} \frac{\partial x^*}{\partial \bar{\alpha}_m} = \frac{\partial \Pi_s(\bar{\alpha}_m, x^*)}{\partial \bar{\alpha}_m}.$$

This is due to $\frac{\partial \Pi_s(\bar{\alpha}_m, x)}{\partial x} \Big|_{x=x^*} = 0$ because x^* solves the first order condition of the owner's objective function. Therefore,

$$\frac{d\Pi_R^*}{d\bar{\alpha}_m} = \begin{cases} K - x_1 & \text{if } \bar{\alpha}_m < \alpha_l, \\ K - \tilde{x}_R & \text{if } \bar{\alpha}_m \in [\alpha_l, \alpha_h], \\ K & \text{if } \bar{\alpha}_m > \alpha_h, \end{cases} \quad (1.34)$$

and

$$\frac{d\Pi_{NR}^*}{d\bar{\alpha}_m} = \begin{cases} K - x_4^* & \text{if } \bar{\alpha}_m \in [0, \alpha_3], \\ K - x_3^* & \text{if } \bar{\alpha}_m \in (\alpha_3, \alpha_2], \\ K - x_2 & \text{if } \bar{\alpha}_m > \alpha_2. \end{cases} \quad (1.35)$$

In addition, one can show that $x_4^* > x_3^* > x_2$, $x_1 > \tilde{x}_R$, and $x_2 > x_1$. Also note that $K - x_4^* > 0$ because we assumed that the owner has sufficient capital to

procure the optimal order quantity for a pure equity newsvendor. Therefore, $\frac{d\Pi_R^*}{d\bar{\alpha}_m} > \frac{d\Pi_{NR}^*}{d\bar{\alpha}_m} \geq 0$, which implies that if $\Pi_R^*(0) > \Pi_{NR}^*(0)$ then investing with risk is optimal for all $\bar{\alpha}_m \geq 0$ because the two functions never intersect.

For a large $\bar{\alpha}_m$ value, investing with bankruptcy risk option leads to no investment in the newsvendor (i.e., $x_1^* = 0$). However, the owner invests x_2 if she chooses to invest without bankruptcy risk. Therefore,

$$\begin{aligned}\Pi_{NR}^* - \Pi_R^* &= (1 + \bar{\alpha}_m)(K - x_2) + (1 + \alpha)x_2 - ((1 + \alpha)c - s)q^{BWO}(x_2) \\ &\quad + (p - s) \left[\int_0^{q^{BWO}(x_2)} \bar{F}(\xi) d\xi - \tau \int_{d_T(q^{BWO}(x_2), x_2)}^{q^{BWO}(x_2)} \bar{F}(\xi) d\xi \right] \\ &\quad - \left((1 + \bar{\alpha}_m)K + (p - s) \int_{\frac{(1+\alpha)c-s}{p-s}\theta}^{\theta} \bar{F}(\xi) d\xi \right) \\ &= -(1 + \bar{\alpha}_m)x_2 + C_2\end{aligned}$$

where C_2 is a constant. Hence, $\Pi_R^* > \Pi_{NR}^*$ for a sufficiently large $\bar{\alpha}_m$ value. Combining this result with $\frac{d\Pi_R^*}{d\bar{\alpha}_m} > \frac{d\Pi_{NR}^*}{d\bar{\alpha}_m} \geq 0$ implies that if $\Pi_R^*(0) \leq \Pi_{NR}^*(0)$ then there must exist a unique $\tilde{\alpha} \geq 0$ such that $\Pi_R^*(\tilde{\alpha}) = \Pi_{NR}^*(\tilde{\alpha})$. Hence, investing without bankruptcy risk is optimal when $\bar{\alpha}_m \leq \tilde{\alpha}$, and investing with bankruptcy risk is optimal otherwise. \square

For completeness, it may be noted that the above solutions continue to hold if the credit limit is never binding, i.e., in the scenario shown in Figure 1.2(b). When $q^L(0) \geq q^{BWR}(0)$, the bank's credit limit is greater than the newsvendor's optimal purchase quantity for $x \in (0, x_2)$. Thus, Case 1, i.e., borrowing with risk and ordering the bank's optimal quantity, does not arise because the credit limit is never binding. Further, from Lemma 1.3.3, the owner's payoff function is convex in Case 2. Therefore, the optimal solution for the borrowing with risk scenarios is either at $x = 0$ or $x = x_2$. The optimal solution for borrowing without bankruptcy risk, Proposition 1.3.5, remains unchanged because Cases 3-5 are

unaffected by the credit limit. Combining these together, the owner's global optimal solution when $q^L(0) \geq q^{BWR}(0)$ is at either $x^* = 0$ or $x^* = x_{NR}^*$. Writing the owner's payoff functions for $x = 0$ and $x = x_{NR}^*$ and comparing them, we again obtain a threshold value such that $x = 0$ is optimal for $\bar{\alpha}_m$ values that exceed the threshold and $x = x_{NR}^*$ is optimal for $\bar{\alpha}_m$ values that are below the threshold. We omit this step because it is analogous to Proposition 1.3.6.

In general, the value of $\tilde{\alpha}$ can be determined by a numerical search technique to find the intersection point between $\Pi_R^*(\bar{\alpha}_m)$ and $\Pi_{NR}^*(\bar{\alpha}_m)$.

Example 1.3.1 ($\tau = 0$) *Table 1.1 presents all possible equilibrium outcomes in the simplified case with zero taxes. The solution can be a pure equity firm, one with both debt and equity, or a pure debt firm. The equilibrium order quantity equals \tilde{q}_{NR} , \tilde{q}_R or θ , respectively, in the three cases, where $\tilde{q}_{NR} \geq \tilde{q}_R \geq \theta$. Which of these three cases arises at equilibrium depends on market conditions through $\bar{\alpha}_m, \alpha$ and the bank's belief. Observe that the overage and underage costs of the classical newsvendor model are adjusted to capture market conditions. For example, $\bar{\alpha}_m$ can be interpreted as the cost of capital of a pure equity firm, and is incorporated in \tilde{q}_{NR} because the cost of purchasing one unit equals $(1 + \bar{\alpha}_m)c$. The borrowing interest α is additionally incorporated in the formulas for \tilde{q}_R and θ . As per Proposition 1.3.6, any of the three cases in Table 1.1 can occur when $\tilde{\alpha}$ exits, but the pure equity case does not occur otherwise.*

The main financial implication of this example is that a debt-equity mix or a pure debt firm can occur at equilibrium even in the absence of taxes. This result differs from the rationale behind the tradeoff theory originating from Kraus and Litzenberger [48], in which the optimal capital structure balances a trade-off between tax benefits of debt and cost of bankruptcy. Taxation induces the

Table 1.1: Possible equilibrium values of inventory, debt, and equity in the absence of taxation when $\tilde{\alpha}$ exists (i.e., when Π_R^* and Π_{NR}^* intersect). If Π_R^* and Π_{NR}^* never intersect then the equilibrium outcome is debt equity mix for $0 \leq \bar{\alpha}_m \leq \alpha_h$ and pure debt for $\bar{\alpha}_m > \alpha_h$. The values of β and θ are defined in Proposition 1.3.3. \tilde{q}_{NR} and \tilde{q}_R can be obtained by setting $\tau = 0$ in Propositions 1.3.4 and 1.3.5. Borrowing without risk cannot arise as an equilibrium outcome when $\tau = 0$.

Possible Case	Order Quantity (q^*)	Equity (x^*)	Debt (w^*)
Pure equity, $\bar{\alpha}_m \in [0, \tilde{\alpha})$	$\tilde{q}_{NR} = \bar{F}^{-1} \left(\frac{(1+\bar{\alpha}_m)c-s}{p-s} \right)$	$c\tilde{q}_{NR}$	0
Debt equity mix, $\bar{\alpha}_m \in (\tilde{\alpha}, \alpha_h]$	$\tilde{q}_R = \bar{F}^{-1} \left(\frac{(1+\bar{\alpha}_m)(c-\frac{s}{1+\alpha})}{p-s} \right)$	$\frac{\tilde{q}_R - \theta}{\beta}$	$\frac{s}{1+\alpha}\tilde{q}_R + \frac{\theta}{\beta}$
Pure debt, $\bar{\alpha}_m > \alpha_h$	θ	0	$c\theta$

firm to borrow because interest expense is tax deductible, but excessive borrowing leads to financial distress. In contrast, our example shows that if the lending terms offered by the bank (i.e., the interest rate and the credit limit) are favorable then the firm might borrow even in the absence of taxation because borrowing increases the expected return to the owner and shifts the risk of bankruptcy to the bank. Conversely, if the lending terms are not favorable to the owner then the firm might choose to use pure equity even in the absence of bankruptcy costs. (In our model, bankruptcy costs are embedded in θ , and our analysis remain valid when $b = 0$.) Thus, taxation and bankruptcy costs are of secondary importance for the firm's capital structure compared to market conditions and firm characteristics (e.g., demand distribution, profitability), both of which shape the firm's borrowing ability and the owner's operational and financial decisions.

1.4 Managerial Implications

1.4.1 Information Asymmetry

The bank's view of the newsvendor's demand may be optimistic or pessimistic compared to the true distribution. If F has first order stochastic dominance over G , then the bank is pessimistic, otherwise it is optimistic. As the bank gets more pessimistic, it tightens the credit limit as shown in the credit limit formula (1.19). Credit tightening decreases the firm's leverage as expected. If this were the only consideration, we would expect the order quantity to be decreasing in the bank's degree of pessimism. However, the degree of pessimism also affects the owner's equity investment decision through $\tilde{\alpha}$ and the credit limit. As a result, the optimal order quantity need not always decrease as the bank gets more pessimistic.

Table 1.2 illustrates this outcome by showing the equilibrium order quantity in two scenarios, (i) when the bank is optimistic and (ii) when the bank is pessimistic. We vary $\bar{\alpha}_m$ and keep all other parameters identical between the two scenarios in order to illustrate the outcomes. In reality, $\bar{\alpha}_m$ may be correlated with other model parameters. Since $\tilde{\alpha}$ varies with the degree of optimism, we find that $\tilde{\alpha}$ equals 8.5% and 11.1% in the games with the optimistic and the pessimistic banks, respectively. When $\bar{\alpha}_m$ is smaller than 8.5%, the owner constitutes the newsvendor as a pure equity firm. Therefore, information asymmetry has no effect on operational investment because the equilibrium order quantity is the same regardless of whether the bank is optimistic or pessimistic.

As $\bar{\alpha}_m$ increases, it exceeds the value of $\tilde{\alpha}$ for the optimistic bank. Therefore,

Table 1.2: Equilibrium order quantity and Debt to Assets (DA) ratio as a function of the expected market return. Demand is exponential with mean 10. Optimistic, and pessimistic banks believe that demand is exponential with mean 12 and 8, respectively. $p = 1, c = 0.6, b = 0.2, s = 0.1, \alpha = 0.15, \tau = 0$. When the bank is pessimistic, the owner chooses pure equity financing if $0 \leq \bar{\alpha}_m \leq 11.1\%$, debt-equity mix if $11.1\% < \bar{\alpha}_m \leq 42.0\%$, and pure debt financing if $\bar{\alpha}_m > 42.0\%$. When the bank is optimistic, the owner chooses pure equity financing if $0 \leq \bar{\alpha}_m \leq 8.5\%$, debt-equity mix if $8.5\% < \bar{\alpha}_m \leq 32.0\%$, and pure debt financing if $\bar{\alpha}_m > 32.0\%$.

$\bar{\alpha}_m$	Pessimistic Bank		Optimistic Bank	
	Order Quantity	DA Ratio	Order Quantity	DA Ratio
5%	5.30	0	5.30	0
10%	4.75	0	4.51	0.61
25%	3.06	0.60	3.06	0.83
50%	1.62	1	2.44	1

when faced with an optimistic bank, the owner finds it attractive to contribute less equity and borrow with bankruptcy risk. Whereas when faced with a pessimistic bank, the owner continues to contribute more equity and have zero borrowing. As a result, for $\bar{\alpha}_m \in [0.085, 0.111)$, the equilibrium order quantity is higher when the owner faces a pessimistic bank.

As $\bar{\alpha}_m$ exceeds 11.1%, the newsvendor borrows up to the credit limit regardless of the bank's view. The owner compensates for the tighter credit limit given by the pessimistic bank by contributing more equity in the newsvendor. As a result, the equilibrium order quantity is unaffected by information asymmetry even though financial leverage varies.

In summary, information asymmetry leads to some counterintuitive findings. There is a range of values of $\bar{\alpha}_m$ (i.e., $8.5\% < \bar{\alpha}_m \leq 11.1\%$) for which an optimistic bank leads to a lower order quantity because the owner finds it attractive to contribute less equity. In a higher range of values of $\bar{\alpha}_m$ (i.e., $11.1\% < \bar{\alpha}_m \leq 32.0\%$), information asymmetry affects leverage, but has no effect

on the order quantity because tighter credit environment provided by a pessimistic bank is compensated by more equity. Finally, when $\bar{\alpha}_m$ is very high (e.g., 50%), tighter credit due to information asymmetry depresses operational investment.

1.4.2 Probability of Bankruptcy

Our model shows that the probability of bankruptcy depends on the parameters of the newsvendor model, the attractiveness of the external investment alternative, and information asymmetry. At equilibrium,

$$Pr(\text{Bankruptcy}) = \begin{cases} 0 & \text{if } \tilde{\alpha} \text{ exists and } \bar{\alpha}_m \leq \tilde{\alpha}, \\ F\left(\tilde{G}^{-1}\left[1 - \frac{\alpha c}{(1+\alpha)c-s}\left(\frac{p-s}{p-s+b}\right)\right]\right) & \text{if } \tilde{\alpha} \text{ does not exist or } \bar{\alpha}_m > \tilde{\alpha}. \end{cases} \quad (1.36)$$

To see this, note that if the firm borrows with risk, then the credit limit is *always* binding. Thus, bankruptcy occurs if the demand is less than $d_B(q^L(x), x)$. The probability of occurrence of this event is $Pr(\xi \leq d_B(\beta x + \theta, x)) = F\left(\tilde{G}^{-1}\left[1 - \frac{\alpha c}{(1+\alpha)c-s}\left(\frac{p-s}{p-s+b}\right)\right]\right)$.

This characterization shows that the equilibrium probability of bankruptcy is either zero or a fixed scalar depending on the regime in which the equilibrium solution lies. The fixed scalar arises because the bank's profit function in (1.19) is a newsvendor-like formula. That is, similar to the newsvendor model, the optimal solution for the bank fixes the tail probability of demand. In Section 1.6.2, we show that a similar result in which the probability of bankruptcy is either zero or a positive scalar also arises when the bank optimizes the interest rate without imposing a credit limit. Hence, this result is due to the bank's ability to influence the owner's equity investment decision, not due to the bank's lending

model.

The bankruptcy formula in (1.36) suggests hypotheses regarding the drivers of bankruptcy risk at equilibrium. The effect of information asymmetry is captured by the term $F(\bar{G}^{-1}(\cdot))$. It implies that there is a lower probability of bankruptcy if the bank is pessimistic about the demand distribution. This occurs because the credit limit is smaller. Moreover, the value of $\tilde{\alpha}$ is larger so that there is a smaller range of values of $\bar{\alpha}_m$ for which bankruptcy can occur. Conversely, the bankruptcy probability is higher when the bank is optimistic. Besides information asymmetry, we observe that the probability of bankruptcy increases in p , s/c , and α because the bank is willing to allow the newsvendor to have a higher order quantity as p , s/c or α increases. It decreases in b because the bank is more conservative when the bankruptcy cost is high.

Additionally, the market return $\bar{\alpha}_m$ and the threshold $\tilde{\alpha}$ affect the bankruptcy probability by determining whether the owner will inject enough equity to avoid borrowing with risk. Holding all other parameters constant, a higher expected return in the market can increase the probability of bankruptcy from 0 to $F\left(\bar{G}^{-1}\left[1 - \frac{\alpha c}{(1+\alpha)c-s} \left(\frac{p-s}{p-s+b}\right)\right]\right)$ because the owner has other attractive investment alternatives.

1.4.3 Collateral Value of Inventory

Linking the credit limit to inventory enables the bank to secure the loan and ensure that it is used for the intended purpose of inventory procurement alone. In this section, we derive the collateral value of inventory ρ_q that the bank can use to implement ABL. According to Proposition 1.3.3, the bank sets the maxi-

mum order quantity equal to $q^L(x) = \beta x + \theta$. This formula allows us to write the credit limit in an alternative form, i.e., $\psi = \rho_q c q$, because the excess cash will be zero when the firm uses its entire credit line. Using the expressions for β and θ from Proposition 1.3.3, we find that the bank can limit the newsvendor to order a maximum of $q^L(x)$ by setting $\rho_q = \frac{c q^L(x) - x}{c q^L(x)}$. Writing ρ_q as a function of x gives $\rho_q = 1 - x / \left(\left(\frac{(1+\alpha)c}{(1+\alpha)c - s} \right) x + c\theta \right)$. Thus, the optimal collateral value of inventory depends on the newsvendor parameters, the bank's belief, and equity. Interestingly, ρ_q decreases in x because offering a high credit limit to a firm that already has high equity could allow the firm to order an excessive amount, which could make inventory liquidation more likely. The bank prevents this undesirable outcome by lowering ρ_q . It is also worthwhile to note that ρ_q decreases in θ . That is, a pessimistic bank uses a lower ρ_q value, which leads to a tighter credit limit.

In practice, asset based lenders use simple rules of thumb, such as historical salvage value, to determine ρ_q . Our model shows that they can improve their practice by tailoring the collateral value of inventory to a firm using objective optimization criteria. Moreover, they do not need to disclose the details of these criteria to a potential borrower because specifying α and ρ_q will be sufficient to offer an asset based loan secured solely by inventory.

Consider the example presented in Table 1.2. The salvage value s is 0.1 and the purchase cost c is 0.6 for both the optimistic and the pessimistic banks. Thus, the rule of thumb gives $\rho_q = s/c = 0.167$ (i.e., setting the credit limit $\psi = sq$). However, the optimal ρ_q varies due to the differences in the banks' beliefs. For instance, when $\bar{\alpha}_m = 0.10$, we find that the pessimistic bank sets ρ_q equal to 0.39 and the equilibrium outcome is $(q, w, x) = (4.75, 0, 2.85)$, whereas the optimistic bank sets it equal to 0.61 and the equilibrium outcome

is $(q, w, x) = (4.51, 1.65, 1.06)$. The main difference between optimizing the credit limit and determining it using the rule of thumb arises due to the way inventory is assessed. While the asset based loan is based on the starting inventory, the amount of inventory that will have to be liquidated in case of a bankruptcy is random and depends on the realized demand. Our model takes this into account through the bank's belief, whereas the rule of thumb omits the randomness in demand.

1.4.4 Capital Structure at Equilibrium

Through the interaction of ABL and the newsvendor model, our analysis leads to empirically testable predictions regarding the relationship between a firm's operational characteristics and its capital structure. We present three such predictions. Future empirical research can examine the extent to which capital structure decisions can be explained by operational dynamics.

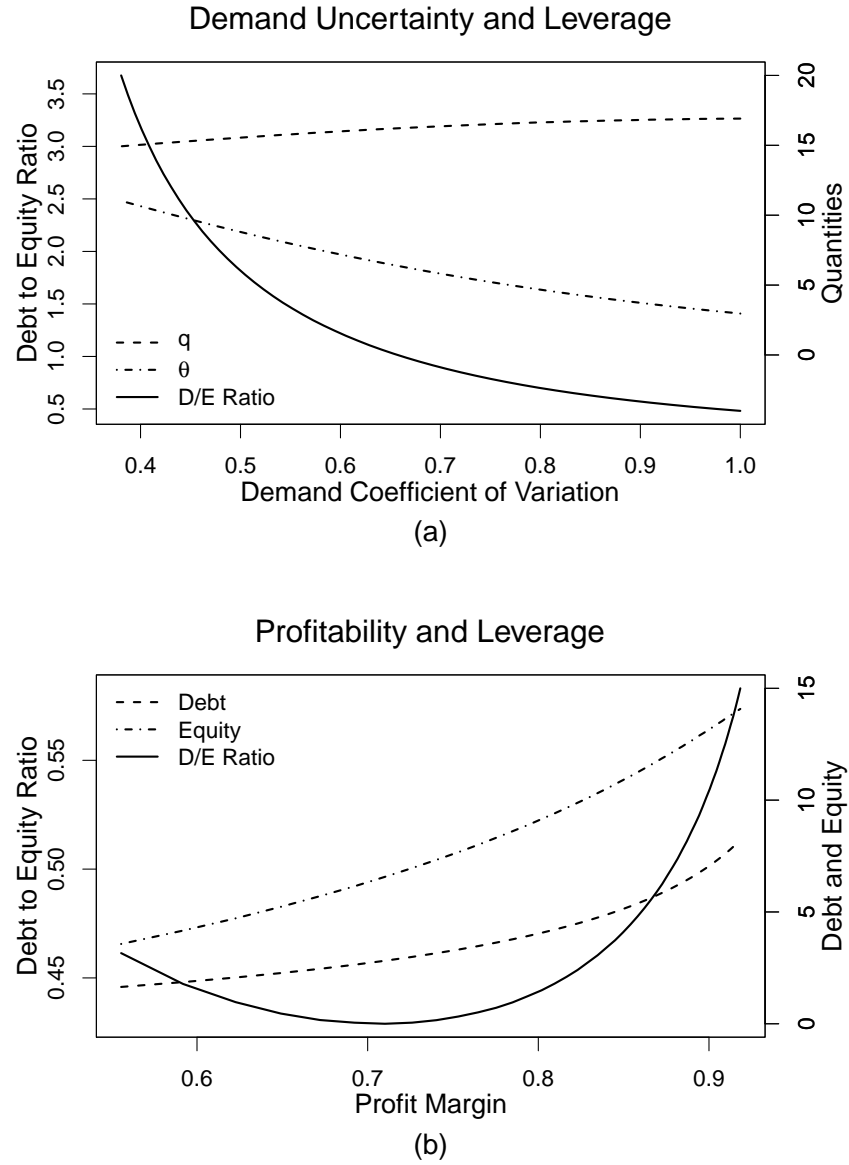
1. *Leverage is negatively correlated with demand uncertainty.* The newsvendor model enables us to model the relationship between demand uncertainty and leverage. Demand uncertainty can be measured by the coefficient of variation of demand. When the profit margin is sufficiently high, the equilibrium order quantity increases in demand uncertainty due to a rise in the safety stock. On the other hand, the minimum credit line offered by the bank declines because the bank wants to limit its losses, which are more likely when the demand is more uncertain. An increase in the equilibrium order quantity and a decrease in the minimum credit line imply an increase in the equity investment. Thus, the debt-equity ratio declines. Figure 1.3(a) illustrates this reasoning.

To the best of our knowledge, no papers use demand uncertainty as a proxy for operational risk within a capital structure framework, but empirical evidence generated by other risk proxies suggests that leverage generally has a negative correlation with risk (Myers [64]). Despite this evidence, the relationship between risk and leverage is ambiguous in the tradeoff theory and the pecking order theory models (Frank and Goyal [35] and references therein). Hence, our model presents a theoretical link between operational risk and leverage that may help address this gap.

2. *There is a non-monotone relationship between profitability and leverage.* The equilibrium order quantity increases in the after tax profit margin because the newsvendor provides a more attractive investment opportunity. Meanwhile, both the owner's tendency to inject equity and the bank's tendency to provide a loan increase in the newsvendor's profitability. As shown in Figure 1.3(b), these dynamics create a U-shaped relationship between profitability and the debt-equity ratio. This result arises because debt and equity increase at different rates, which makes their ratio non-monotone.

The majority of the tradeoff theory models predict a positive correlation between profitability and leverage, whereas the pecking order theory models predict a negative correlation (Frank and Goyal [35]). Both predictions are at odds with the empirical evidence. For example, the empirical findings of Xu and Birge [11] verify the existence of a U-shaped relation between profitability and leverage when the profit margin is sufficiently high. Similar to our work, Birge and Xu provide an operational justification for this observation. They also discuss the shortcomings of the related empirical studies that seek to analyze a seemingly non-monotone relationship using linear regression.

Figure 1.3: The equilibrium capital structure as a function of demand uncertainty and profitability. $\bar{\alpha}_m = 0.15$, $\alpha = 0.10$, $p = 6$, $c = 1$, $s = 0.2$, $b = 0.2$, and $\tau = 0.3$. In Figure (a), demand is Weibull with $E_F[\xi] = 10$, and the bank believes that it is Weibull with $E_G[\xi] = 9$. We vary the shape and scale parameters of F and G to measure the impact of demand variability. In Figure (b), demand is exponential with $E_F[\xi] = 10$, and the bank believes that it is exponential with $E_G[\xi] = 9$. We vary p between 4 and 8 to measure the impact of firm profitability. We define the profit margin as $(p - c)/(p - s)$.



3. *The newsvendor model can explain intra-industry differences in leverage.* The empirical corporate finance literature shows that capital structure varies both across industries and across firms within the same industry (Leary and Graham [51]). In fact, a substantial amount of unexplained variation in capital structure is firm specific and time invariant (Lemmon et al. [55]). However, the trade-off theory and the pecking order theory models cannot accurately explain firm specific differences in capital structure (Myers [64] and Frank and Goyal [35], respectively). Our model provides a theoretical framework to justify and analyze these differences. For example, the newsvendor model predicts a positive correlation between inventory liquidation value and leverage. s/c in the newsvendor model captures the relative liquidation value of the firm's inventory. The asset based credit limit increases in s/c by (1.19). That is, tangible assets provide more debt capacity, which leads to a high debt-equity ratio for a borrower firm. This prediction is consistent with the empirical evidence suggesting that leverage has a positive correlation with the firm's assets' liquidation value (Harris and Raviv [40] and references therein).

To sum up, our model's ability to capture the details of operational and financial dynamics in a realistic framework sheds light on some empirical observations that cannot be accurately explained by the mainstream theoretical models. This implication is well aligned with the recent studies showing the superiority of equilibrium models in explaining capital structure choice (e.g., MacKay and Phillips [58] and references therein).

1.5 Conclusions

We characterize the equilibrium order quantity, probability of bankruptcy, and capital structure of a firm in a game played between a small business owner and an asset based lender. Our results are driven by the economic considerations of the risk of bankruptcy, the expected return to the owner of the firm, and the credit limit imposed by the bank. Taxation and cost of bankruptcy play a secondary role. In fact, all of our results go through under zero taxes and zero cost of bankruptcy. Moreover, the form of the probability of bankruptcy is the same even without information asymmetry or under interest rate optimization (as shown in Section 1.6).

Our study falls in a rich area of research. Issues examined in this chapter can be studied under alternative models, and some aspects of our model can be generalized in future research. For example, players, esp. the owner, may be modeled as expected utility maximizers, and agency issues can be added. It may also be productive to allow trading between the bank and the owner or to replace them with two investor classes. There may be competition among banks, which can affect the equilibrium outcome under information asymmetry. The bank's strategic interactions with the owner can also be modeled. Section 1.6 provides a starting point for such an analysis by the deriving the optimal lending interest rate, which is a function of $\bar{\alpha}_m$. Another extension is to introduce a joint distribution for ξ and $\bar{\alpha}_m$ to capture the correlation between consumption and macroeconomic conditions. Signaling and firm-bank coordination through a menu of contracts can also be considered. For example, the bank may consider using a signaling mechanism to infer the true demand distribution of a potential borrower. However, designing such a mechanism is not straightforward

because firms with bad demand prospects may replicate good firms' actions. As a result, a pooling equilibrium may exist in which order quantities and borrowing amounts become uninformative; see Greenwald and Stiglitz [39] (page 37) for a similar argument. Another potential direction is to extend our model to a multi-period setting in order to capture time-varying bankruptcy risk. Finally, future empirical research may examine the predictions emerging from our study regarding the links between operational parameters, capital structure and the probability of bankruptcy.

1.6 Model Extension: Interest Rate Optimization

Our assumption of fixed interest rate prompts two questions: (i) why does the bank not optimize the interest rate for an individual borrower firm, and (ii) is our result on probability of bankruptcy dependent on the assumption of fixed interest rate. In Section 1.1, we showed that interest rates are capped for some loans in the economy, and for such loans, the assumption of fixed interest rates is appropriate. In addition, our work differs from the literature in operations wherein banks optimize interest rates or wherein the interest rate is set to achieve the risk-free rate of return on average because equity is a decision variable and there is information asymmetry.

The literature in economics shows that at any given interest rate, under information asymmetry, firms with bad projects (i.e., poor demand prospects in our model) will have a tendency to over-borrow compared to firms with good projects. This creates the well known adverse selection problem for the bank because increasing the interest rate can increase the riskiness of the bank's loan

portfolio by changing the set of borrowing firms and inducing firms to make riskier investments (Stiglitz and Weiss [77]). (A riskier investment corresponds to a higher order quantity in our model.) As a result, credit rationing, under which some borrowers receive smaller loans than they demand, may exist because it may not be optimal for the bank to increase the interest rate until loan supply equals loan demand. Greenwald and Stiglitz [39] (page 15) explain this market failure (i.e., supply-demand mismatch) by stating that “in credit markets, it is by now well established that lenders who are less well-informed than borrowers about the risk characteristics of the borrower’s investment projects may well respond by fixing interest rates and (under certain conditions) rationing credit.”

The implication of adverse selection for our model is that the lending interest rate should be optimized for a *population of borrowers*, not for an individual firm. Such a population can be defined based on model primitives, including G , p , c , and s . Therefore, if the bank has the flexibility to optimize the interest rate, then the optimal interest rate will be a function of model primitives, but will not depend on the owner’s actions. Hence, α continues to be exogenous for a given firm. This result arises because making α a function of x , w and/or q amplifies adverse selection by urging firms with good (bad) demand prospects to borrow less (more).

This section explores these implications in detail.

1. We show that increasing the interest rate does not help the bank because it leads to an equilibrium outcome in which the interest rate is very high and only firms with poor demand prospects borrow.
2. We present the bank’s interest rate optimization problem in the context

of information asymmetry, and validate that the optimal interest rate that maximizes the bank's expected profit under G is a function of model primitives, including $\bar{\alpha}_m$, G , p , and c , but is independent of the owner's and the firm's decisions (i.e., x , w , and q). Thus, the fixed interest rate α in our model can be interpreted as the optimal interest rate for a population of borrowers or an exogenous interest rate imposed by a government agency. This result holds regardless of whether the bank imposes a credit limit or not.

3. We show numerically that imposing a credit limit increases the bank's true expected profit because it prevents over-borrowing. Moreover, if the bank imposes a credit limit then charging an exogenous interest rate that is lower than the optimal interest rate under G increases the bank's true expected profit. However, if the bank does not impose a credit limit then charging an exogenous interest rate that is lower than the optimal interest rate under G decreases the bank's true expected profit.

We illustrate interest rate optimization using a population of two borrowers. Suppose there are two owners with two proprietary newsvendor type investment opportunities. Firm i faces random demand ξ_i characterized by F_i , $i \in \{1, 2\}$. The bank cannot distinguish among the firms, and believes that both firms face identically distributed demand functions characterized by G . We refer to G as the bank's belief. $\bar{\alpha}_m$, p , c , and K are identical for both owners and common knowledge to the bank. For analytical tractability, we focus on a special case in which the tax rate τ , bankruptcy cost b and salvage value s are zero. Our results continue to apply when these quantities are positive.

1.6.1 Interest Rate Optimization with A Credit Limit

In this section, we show how the bank can find the optimal α value that maximizes its expected profits under its belief G given the credit limit computed in Proposition 1.3.3. Setting $s = b = 0$ and replacing F with G in Example 1.3.1 show that a pure equity (PE) firm facing G orders $q^{PE} = \bar{G}^{-1}\left(\frac{(1+\bar{\alpha}_m)c}{p}\right)$. The corresponding expected profit for the owner under the bank's belief G is

$$\Pi_G^{PE} \equiv (1 + \bar{\alpha}_m)(K - cq^{PE}) + p \int_0^{q^{PE}} \bar{G}(\xi) d\xi. \quad (1.37)$$

Similarly, a firm that uses a mix of debt and equity (DE) orders $q^{DE} = \bar{G}^{-1}\left(\frac{(1+\bar{\alpha}_m)c}{p}\right)$, which is equal to q^{PE} . The corresponding expected profit for the owner under G is equal to

$$\Pi_G^{DE}(\alpha) \equiv (1 + \bar{\alpha}_m)(K - c(q^{PD} - \theta)) + p \int_{\frac{(1+\alpha)c}{p}\theta}^{q^{PD}} \bar{G}(\xi) d\xi, \quad (1.38)$$

where $\theta = \frac{p}{(1+\alpha)c} \bar{G}^{-1}\left(\frac{1}{1+\alpha}\right)$. Lastly, a pure debt (PD) firm orders θ , and the owner's expected profit under G is

$$\Pi_G^{PD}(\alpha) \equiv (1 + \bar{\alpha}_m)K + p \int_{\frac{(1+\alpha)c}{p}\theta}^{\theta} \bar{G}(\xi) d\xi. \quad (1.39)$$

Under the bank's belief G , the firm should borrow if α is such that $\max\{\Pi_G^{PD}(\alpha), \Pi_G^{DE}(\alpha)\} \geq \Pi_G^{PE}$. This inequality is a participation constraint, which ensures that the owner is better off by using pure debt financing or a mix of debt and equity rather than pure equity financing. Following Proposition 1.3.3 and Example 1.3.1, if the firm borrows, the bank's expected profit under G will be equal to $p \int_0^{\frac{(1+\alpha)c}{p}\theta} \bar{G}(\xi) d\xi - c\theta$. Hence, the bank solves the following interest rate optimization problem:

$$\begin{aligned} \max_{\alpha \geq 0} \quad & p \int_0^{\frac{(1+\alpha)c}{p}\theta} \bar{G}(\xi) d\xi - c\theta \\ \text{s.t.} \quad & \max\{\Pi_G^{PD}(\alpha), \Pi_G^{DE}(\alpha)\} \geq \Pi_G^{PE}. \end{aligned}$$

We observe that the optimal interest rate under a credit limit α_C^* is a function of model primitives, including G , because the true distribution \bar{F}_i , actual order quantity or equity investment do not appear in the above optimization problem. Hence, the optimal interest rate $\alpha_C^*(p, c, \bar{\alpha}_m, G)$ is exogenously determined for a potential borrower.

When salvage value, taxation, and bankruptcy costs are added to the model, the equilibrium order quantity and starting equity become implicit functions of the interest rate. The bank can find the firm's best response, which consists of an order quantity $q(\alpha; G)$ and a starting equity $x(\alpha; G)$. In fact, replacing \bar{F} terms with \bar{G} in Propositions 1.3.4 and 1.3.5 gives the potential equilibrium outcomes under the bank's belief for a given α . Once the bank computes $x(\alpha; G)$ and $q(\alpha; G)$ for a continuum of α values, it can find the optimal α value that maximizes its expected profits. Despite being notationally cumbersome, this optimization exercise still has a single decision variable α . Hence, α_C^* is still exogenously determined, but becomes a function of additional model primitives, including τ , b , and s . The optimal solution under taxation, bankruptcy costs, and a positive salvage value, $\alpha_C^*(p, c, s, \tau, b, \bar{\alpha}_m, G)$, corresponds to the optimal interest rate for the bank's problem we present in Section 1.3.

1.6.2 Interest Rate Optimization without A Credit Limit

Another lending model for the bank is unsecured lending without a credit limit, as studied by Dada and Hu [26]. The interest rate is the only decision variable for the bank.

The newsvendor-bank interaction in this model is similar to the case that we

illustrate in Figure 1.2(b). That is, for a given starting equity x and interest rate α , the firm's order quantity is not restricted by a credit limit. The absence of the credit limit implies that if $x \in [0, x_2]$ then the firm borrows with risk, and the order quantity solves (1.7). Furthermore, as we show in Lemma 1.3.3, the owner's profit function is convex in the interval in which the firm borrows with risk without a binding credit limit. This convexity result implies that in the absence of a credit limit, if the owner decides to borrow then it is optimal to create a pure debt firm.

Continuing the special case with $\tau = 0$ and $s = b = 0$, the bank believes that if an owner chooses to borrow, she creates a pure debt firm, and the order quantity solves

$$\tilde{q}^{PD} = \bar{G}^{-1} \left(\frac{(1 + \alpha)c}{p} \bar{G} \left(\frac{(1 + \alpha)c}{p} \tilde{q}^{PD} \right) \right). \quad (1.40)$$

This result follows from setting τ, b, s equal to zero in Proposition 1. As a result, the owner's expected ending cash position under G is

$$\tilde{\Pi}_G^{PD}(\alpha) \equiv (1 + \bar{\alpha}_m)K + p \int_{\frac{(1+\alpha)c}{p} \tilde{q}^{PD}}^{\tilde{q}^{PD}} \bar{G}(\xi) d\xi. \quad (1.41)$$

Another alternative for the owner is to create a pure equity firm. This case is identical to the pure equity case presented under a credit limit. That is, a firm facing G orders $\tilde{q}^{PE} = \bar{G}^{-1} \left(\frac{(1 + \bar{\alpha}_m)c}{p} \right)$, and the owner's expected ending cash position is

$$\tilde{\Pi}_G^{PE} \equiv (1 + \bar{\alpha}_m)(K - \tilde{q}^{PE}) + p \int_0^{\tilde{q}^{PE}} \bar{G}(\xi) d\xi. \quad (1.42)$$

The bank believes that the owner creates a pure debt firm if $\tilde{\Pi}_G^{PD}(\alpha) \geq \tilde{\Pi}_G^{PE}$. Otherwise, she creates a pure equity firm. If the owner creates a pure debt firm, then the bank's expected profit under G will be equal to $p \int_0^{\frac{(1+\alpha)c}{p} \tilde{q}^{PD}} \bar{G}(\xi) d\xi - c\tilde{q}^{PD}$. Hence, the bank solves the following interest rate optimization problem:

$$\max_{\alpha \geq 0} \quad p \int_0^{\frac{(1+\alpha)c}{p} \tilde{q}^{PD}} \bar{G}(\xi) d\xi - c\tilde{q}^{PD}$$

$$\text{s.t.} \quad \tilde{\Pi}_G^{PD}(\alpha) \geq \tilde{\Pi}_G^{PE}.$$

Similar to the interest rate optimization with a credit limit, this problem has a single decision variable α , and the optimal interest rate is a function of model primitives, including G . Hence, the optimal interest rate under unconstrained borrowing α_U^* is exogenously determined for a potential borrower.

The equilibrium probability of bankruptcy of a firm i is either zero or a positive scalar because firm i is either a pure equity or a pure debt firm. If it is a pure equity firm, then the equilibrium order quantity \tilde{q}_i^{PD} solves

$$\tilde{q}_i^{PD} = \bar{F}_i^{-1} \left(\frac{(1 + \alpha_U^*)c}{p} \bar{F}_i \left(\frac{(1 + \alpha_U^*)c}{p} \tilde{q}_i^{PD} \right) \right). \quad (1.43)$$

Hence, the probability of bankruptcy is equal to $F \left(\frac{(1 + \alpha_U^*)c}{p} \tilde{q}_i^{PD} \right)$. Obtaining a fixed probability of bankruptcy result under interest rate optimization without a credit limit shows the robustness of the equilibrium probability of bankruptcy result we present in Section 1.4.2.

Once again, adding taxation and bankruptcy costs to the model makes the optimal interest a function of additional model primitives, but α_U^* remains exogenous.

It is worthwhile to note that this optimization procedure is similar to the one presented by Dada and Hu (2008). Dada and Hu find the equilibrium interest rate and order quantity in a game played between a newsvendor firm and a profit maximizing bank. Starting equity is given, and the bank, which has full information about the firm's demand prospects, determines the interest rate that maximizes its expected profit. In their model, the equilibrium interest rate is a function of the firm's starting equity because starting equity is not a

decision variable. Furthermore, adverse selection does not exist because there is no information asymmetry.

1.6.3 Adverse Selection with A Credit Limit

While optimizing the interest rate for a population of borrowers under a credit limit increases the bank's expected profit under its belief, it can still lead to lower profits due to asymmetric information. We illustrate the existence of adverse selection with a numerical example. To be consistent with the adverse selection literature, we assume that

$$\int_0^x \bar{F}_2(\xi) d\xi \leq \int_0^x \bar{G}(\xi) d\xi \leq \int_0^x \bar{F}_1(\xi) d\xi$$

for all $x \geq 0$ with $E_G[\xi] = E_{F_1}[\xi] = E_{F_2}[\xi]$. That is, the bank's belief is a mean preserving spread of firm 1's demand, and firm 2's demand is a mean preserving spread of the bank's belief. This ordering implies that both firms have the same expected demand, but firm 2 has more demand uncertainty. Furthermore, the bank overestimates firm 1's risk, and underestimates firm 2's risk. The use of mean preserving spread for risk assessment originates from Rothschild and Stiglitz [73], and is common in the literature (e.g., Stiglitz and Weiss [77]).

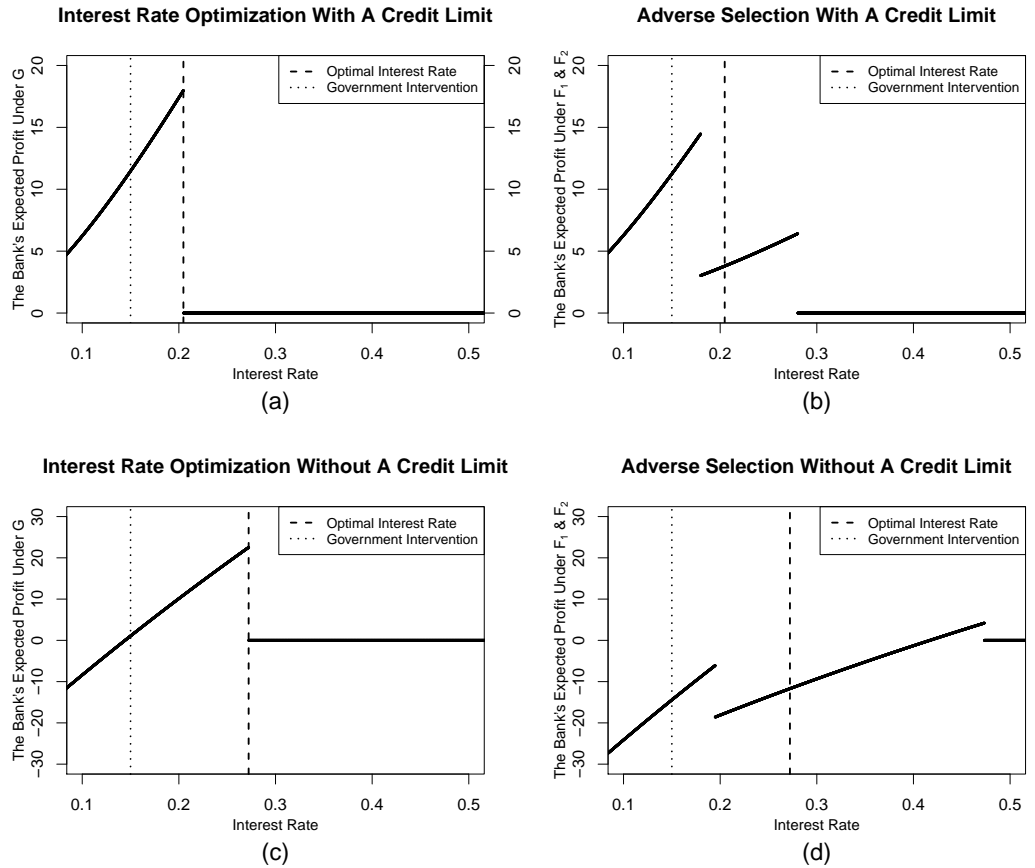
Recall that the bank optimizes the interest rate based on its belief G . For any given α , including $\alpha_C^*(p, c, s, \tau, b, \bar{\alpha}_m, G)$, cost of borrowing is relatively high for firm 1, which has less risky demand prospects, compared to firm 2, which has more risky demand prospects. As a result, owner 1 might choose to create a pure equity firm, whereas firm 2 might be financed with pure debt or debt-equity mix. Figure 1.4 illustrates this well known adverse selection problem. Figure 1.4(a) shows the bank's interest rate optimization problem. The participation

constraint is binding, and $\alpha_C^* = 0.21$ maximizes the bank's expected profits under G . Figure 1.4(b) shows the bank's true expected profit as a function of the interest rate. α_C^* is too high for firm 1. Hence, only firm 2 borrows when $\alpha = \alpha_C^*$, and the bank's expected profit is 3.79. This example illustrates the danger of interest rate optimization under information asymmetry. Namely, the bank loses the opportunity to lend to firm 1 because it sets the interest rate too high. Our example is well aligned with the existing adverse selection literature. For example, Stiglitz and Weiss [77] present a similar result; please see Figure 3 on page 397 in their paper.

A government intervention can help the bank increase its profits. This result arises because a government mandate that forces the bank to charge less than α_C^* can mitigate adverse selection by increasing the quality of borrowers. For instance, in the example we present in Figure 1.4(b), if the government forced the bank to charge no more than 15%, then the bank would charge 15% and both firms would borrow. As a result, the bank's expected profits would be equal to 11.24, which is higher than its expected profits at $\alpha = \alpha_C^*$. Settings similar to this example are used in macroeconomics to justify government control on lending terms, e.g., Ordoover and Weiss [68] and Mankiw [59]. Such government practices are also supported by the data we present in Section 1.1.

Stiglitz and Weiss [77] also show that credit is rationed at equilibrium. In this example, the credit limit serves as a credit rationing mechanism. It is binding for firm 1 if $\alpha \leq 18\%$ and for firm 2 if $\alpha \leq 28\%$. In the next section, we show that interest rate optimization in the absence of a credit limit can have disastrous consequences and that imposing a credit limit mitigates those consequences by preventing over-borrowing.

Figure 1.4: The bank's interest rate optimization with and without a credit limit. There are two firms in the economy. Firm 1's demand is Weibull with mean $\mu_1 = 100$ and parameters $k_1 = 2$ and $\lambda_1 = \mu_1/\Gamma(1 + 1/k_1)$. Firm 2's demand is Weibull with mean $\mu_2 = 100$ and parameters $k_2 = 1$ and $\lambda_2 = \mu_2/\Gamma(1 + 1/k_2)$. The bank believes that both firms have Weibull demand with mean $\mu_B = 100$ and parameters $k_B = 1.5$ and $\lambda_B = \mu_B/\Gamma(1 + 1/k_B)$. $p = 2.5$, $c = 1$, $\bar{\alpha}_m = 0.12$, $\tau = 0$, and $b = s = 0$. Figures (a) and (b) show the bank's interest rate optimization and adverse selection under a credit limit, respectively. In Figure (a), if the bank imposes a credit limit, $\alpha_C^* = 0.21$ is optimal under G . Figure (b) shows adverse selection under a credit limit by showing the bank's expected profits under $F_{i \in \{1,2\}}$ as a function of the interest rate. Firm 1 borrows if $\alpha \leq 18\%$. Firm 2 borrows if $\alpha \leq 28\%$. Figures (c) and (d) show the bank's interest rate optimization and adverse selection without a credit limit, respectively. In Figure (c), if the bank does not impose a credit limit, $\alpha_U^* = 0.27$ is optimal under G . Figure (d) shows adverse selection in the absence of a credit limit by showing the bank's expected profits under $F_{i \in \{1,2\}}$. Firm 1 borrows if $\alpha \leq 19.5\%$. Firm 2 borrows if $\alpha \leq 47.3\%$.



1.6.4 Adverse Selection without A Credit Limit

Interest rate optimization in the absence of a credit limit also suffers from adverse selection. Moreover, an additional complication arises because the bank does not have a credit allocation device to ration credit. Figures 1.4(c) and 1.4(d) show the pitfall of this lending mechanism with a numerical example. We generate Figures 1.4(c) and 1.4(d) with the parameter values we used to generate Figures 1.4(a) and 1.4(b). Figure 1.4(c) shows the bank's interest rate optimization problem. The participation constraint is binding, and $\alpha_U^* = 0.27$ maximizes the bank's expected profits under G . Figure 1.4(d) shows the bank's true expected profit as a function of the interest rate. Once again, α_U^* is too high for firm 1. Hence, only firm 2 borrows when $\alpha = \alpha_U^*$, and the bank's profit is -11.70. The bank loses money in expectation because it cannot prevent firm 2 from over-borrowing. A government regulation that forces the bank to lend at a lower interest rate amplifies over-borrowing. For example, if the bank is forced to charge less than 15%, then it charges 15% with the belief that its expected profit will be equal to 1.03. Both firms borrow at $\alpha = 0.15$. At $\alpha = 0.15$, the true expected profits from firm 1 and firm 2 are 8.38 and -22.82, respectively. Hence, the bank's total expected profit under $F_{i \in \{1,2\}}$ is -14.44. Once again, this result arises because firm 2 borrows excessively, and plays a risk free gamble with the bank's money.

Comparing the true expected profits with and without a credit limit illustrates the benefit of imposing a credit limit. Figure 1.4(d) shows that the bank loses money in expectation if $\alpha \leq 41.7\%$. Put differently, for any given interest rate that is below 41.7%, the bank's true expected profit with a credit limit is higher than its true expected profit in the absence of a credit limit. For ex-

ample, at $\alpha = 15\%$, the bank's true expected profits with and without a credit limit are 11.24 and -14.44, respectively. Hence, imposing a credit limit mitigates adverse selection regardless of whether interest rates are optimized or exogenously determined, whereas using the interest rate as a single instrument can lead to excessive losses.

CHAPTER 2

BALANCING GROWTH AND BANKRUPTCY RISK FOR A CASH CONSTRAINED FIRM

2.1 Introduction

How fast should a firm grow? On the one hand, a company that moves slowly misses profitable investment opportunities. It is eventually outperformed by competitors. On the other hand, a company that tries to grow too fast can run into cash flow problems that lead to bankruptcy. But how slow is too slow, and how fast is too fast? Finding the right answer to this question is crucial to maximizing the long term profits of a firm.

The J. Peterman Co., founded in 1987 as a catalog retailer, is an example of growing too fast. Mr. John Peterman, the founder and first owner of the J. Peterman Co., had initial success in catalog retailing. However, when venture capital firms provided funds for the company, they pressured him to set unrealistic sales targets. An aggressive growth strategy was the only option to achieve these targets in a short time frame. In an effort to grow fast, the company's cash flows got out of control, which led to Chapter 11 bankruptcy in 1999 (Peterman [70]).

Ingres, a data analytics company, has the opposite story. Ingres' main competitor was Oracle in the early 80s. Both companies were small back then. Oracle targeted to grow at a 100% per year rate while Ingres chose a decent but a relatively slower growth target of 50% (Moore [63]). According to Moore, executives at Ingres felt that "[Ingres] simply cannot grow any faster than 50 percent

and still adequately serve customers. No one can. Look at Oracle. They are promising anything and everything and shipping little or nothing... Their customers hate them. They are going to hit the wall." Some 30 years later, Oracle's sales revenue has reached \$35 billion. According to the latest press kit, Ingres' sales revenue was \$68 million in 2008.

These examples illustrate the relationship between growth and the associated risks. A slow growth company avoids risky decisions. Although such a strategy ensures short-term survival, it leads to tiny financial returns. On the other hand, aggressive growth requires making risky investment decisions. If these decisions are successful, then the firm grows at an outstanding pace. Otherwise, the firm fails to survive. Hence, an aggressive growth strategy implies high risk and high return, whereas a conservative growth strategy leads to low risk and low return.

In this chapter, we study the tradeoff between growth and bankruptcy risk. We analyze this tradeoff by setting up a finite horizon cash-constrained inventory model with non-stationary demand. In our model, the firm seeks to maximize its expected ending cash position, discounted at a constant rate. This or a similar target (e.g., maximizing the firm's sales or net worth) is a common objective in practice because most investors (e.g., venture capital firms) seek to exit with a high return on their investment in a short time horizon (e.g., five years). A non-stationary demand stream, which is a function of the firm's past sales, captures demand growth. In addition, a cash constrained setting captures risk because a cash constrained firm may need to borrow to achieve its growth potential.

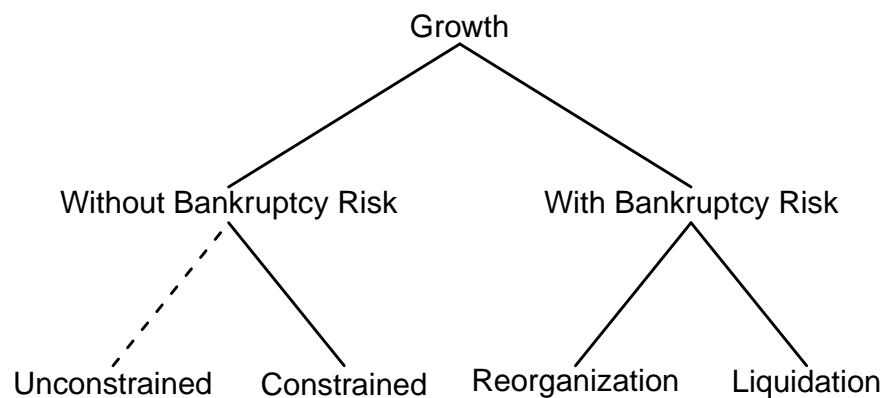
Our primary objective is to identify an optimal or close to optimal inventory

ordering policy for a cash constrained firm. Analyzing an ordering policy reveals the corresponding growth trajectory in our model because the ordering decision is the only operational decision that the firm makes. We first analyze a best case scenario in which the firm can achieve growth without running into cash flow problems. We show the optimality of a modified newsvendor solution that captures the impact of today's order quantity on future demand. Then we introduce financial constraints, and prohibit the firm to borrow with risk. This setting allows us to replicate a self-financing growth trajectory.

After analyzing the above cases, we introduce bankruptcy risk by allowing the firm to borrow from a perfectly competitive lending market. We study both reorganization and liquidation bankruptcies to understand the impact of legal restrictions on the firm's operational and financial decisions. We first focus on a reorganization bankruptcy scenario in which the firm can restructure without any penalties every time it fails to meet its debt obligations. This scenario represents the most favorable bankruptcy case for the firm because it allows the firm to erase its outstanding debt and restart its operations without incurring bankruptcy costs. Our analysis shows that the modified newsvendor solution we obtain under the best case scenario with unconstrained growth is also optimal in this scenario. Hence, we provide an operations management perspective on the benefit of reorganization bankruptcy for entrepreneurial growth. Then we switch our attention to liquidation bankruptcy by studying a stopping time problem in which the firm stops its operations when it fails to meet its debt obligations for the first time. This scenario represents the least favorable bankruptcy case for the firm owner (i.e., the entrepreneur) because she loses her initial investment entirely. Although no closed form solution exists for the optimal order quantity under liquidation bankruptcy, we propose a heuristic solution that bal-

ances the tradeoff between growth and risk. We also perform numerical analysis to show that this heuristic performs well. Figure 2.1 outlines the scenarios we study in this chapter.

Figure 2.1: Different growth scenarios. Unconstrained growth corresponds to the best case scenario in which the firm has sufficient cash to finance its inventory investment. It is a specific instance of constrained growth, which corresponds to financing growth with internal cash flows. Reorganization and liquidation bankruptcy scenarios capture the impact of bankruptcy risk when the firm can use external financing by borrowing from a risk neutral financial market.



Our analysis reveals several managerial insights. First, the optimal order quantity of an unconstrained firm exceeds the classical single-period newsvendor solution. That is, when demand is a function of the firm's past actions, the firm overinvests to fuel growth. Second, comparing the firm's expected ending cash positions under the unconstrained and the constrained cases without bankruptcy risk indicates that internal cash flow constraints (i.e., a self-financing growth strategy) can significantly depress firm growth. Third, our analysis indicates that the firm can achieve a higher expected ending cash position by borrowing with risk, regardless of whether it is reorganized or liquidated in the event of a bankruptcy. Put differently, growth requires making risky investment decisions. Comparing the firm's expected ending cash posi-

tions under liquidation and reorganization bankruptcies shows that the firm can perform better under reorganization bankruptcy. This is not surprising because reorganization bankruptcy gives the firm more leeway to follow a more aggressive growth strategy.

Combining these observations indicates that our model captures the tradeoff between growth and risk. That is, we show that a firm that follows a conservative growth strategy by choosing small order quantities realizes limited growth (e.g., Ingres) whereas a firm that follows a more aggressive growth strategy by choosing large order quantities declares bankruptcy on sample paths with low demand realizations (e.g., The J. Peterman Co.) and enjoys significant growth on sample paths with high demand realizations (e.g., Oracle). Lastly, we numerically examine the probability of bankruptcy, and show that it is higher in the early periods. This result arises because the firm is more likely to face cash flow problems in the early periods due to a combination of low demand and high order quantities. This observation is consistent with practice because most of the firm failures occur in the early stages (Chava and Jarrow [21]).

2.2 Literature Review

Managing growth in the presence of cash flow constraints has received attention in the finance, economics, and strategic management literatures. Empirical evidence indicates that most small firms finance their growth through internal financing (i.e., retained earnings), and, as a result, a small firm's growth is constrained by its ability to generate cash (Carpenter and Petersen [19] and references therein). One of the simplest and most intuitive models that ties a firm's

operational characteristics to its growth prospects is the self-financing growth (SFG) model. This model analyzes the firm's working capital needs and operating expenses as well as its ability to generate profits through sales to compute the maximum SFG rate that can be achieved through internally generated cash flows (e.g., Churchill and Mullins [23]). The SFG model highlights the importance of operations by predicting that an operationally efficient firm can grow faster than an inefficient firm because of its ability to achieve a higher return on investment. However, it relies on very strong assumptions (e.g., no randomness in sales, the firm's past performance is an accurate predictor of its future) to reach this conclusion. We generalize the SFG model in Section 2.3 by formalizing the firm's operational decisions through a multi-period inventory model with random demand.

The firm can relax its internal cash flow constraints by borrowing from financial markets. In fact, Aghion et al. [1] empirically show that access to financial markets enhances firm growth. However, borrowing introduces bankruptcy risk because the firm might fail to repay its debt. From a theoretical standpoint, there are multiple ways to model growth when the firm faces bankruptcy risk. One way is to adopt a mathematical finance approach and assume that the firm's revenues or cash flows follow a Brownian motion. In this approach, the firm declares bankruptcy when its cash reserves hit zero for the first time. Hence, bankruptcy is modeled as a barrier option. For example, Radner and Shepp [71] analyze a firm that seeks to maximize the discounted sum of its expected dividends by selecting a dividend policy and picking a drift-volatility pair from a finite action set, which represents different operational strategies. They characterize the optimal dividend policy, which involves paying no dividends if the cash reserve is less than a threshold and paying out all the excess

cash if the cash reserve exceeds the same threshold, and show that if the firm follows the optimal policy, it goes bankrupt in a finite amount of time with probability one. See Dutta and Radner [31] and Caldentey and Haugh [16] and references therein for other papers that use similar settings to model the relationship between a firm's real (i.e., operational) and financial activities.

Another way to model growth and bankruptcy risk is to assume that there is information asymmetry between the firm and its lenders regarding the firm's investment prospects and/or actions. This assumption leads to borrowing constraints because the lender wants to prevent over-borrowing. The economics literature adopting this framework focuses on the impact of lending contracts and agency issues on a firm's growth prospects and bankruptcy risk (Clementi and Hopenhayn [24] and references therein). Despite providing financial predictions that are consistent with empirical evidence regarding growth and bankruptcy risk, both the mathematical finance approach and the asymmetric information approach lead to limited operational insights due to their highly stylized operational investment models. In contrast, the firm in our model borrows from a perfectly competitive lending market to finance its inventory investment. This setting provides an operations management framework to analyze the relationship between growth and bankruptcy risk.

In the operations management literature, operational decisions under financial constraints and bankruptcy risk have received more attention lately. Among single-period models, Xu and Birge [82], Buzacott and Zhang [15], Dada and Hu [26], and Alan and Gaur [2] analyze a single firm's interactions with financial markets under different modeling assumptions. Similarly, Lai et al. [50], Kouvelis and Zhang [47], and Yang and Birge [84] study the interactions between

supply chain partners in the presence of financial considerations, such as contracting and trade credit. Multi-period inventory models incorporate additional financial dynamics, such as cash flows constraints, wipeout and reorganization bankruptcies, dividend payments, and capital subscriptions, e.g., Li et al. [56], Hu and Sobel [42], Chao et al. [20], and Hu et al. [43]. Financial constraints and bankruptcy risk also affect the firm’s survival strategy (Archibald et al. [6]) and its relations with its supply chain partners (Swinney and Netessine [79], Babich [8]), the choice of production technologies (Lederer and Singhal [53], Boyabatli and Toktay [13]), the optimal time to shut down a firm (Xu and Birge [83]), and the optimal time to offer an IPO (Babich and Sobel [9]). Our work contributes to this stream by modeling the operational and financial decisions of a firm facing a growing demand stream, which is a function of its past actions.

Our work is also related to the capacity management literature, part of which deals with capacity expansion to satisfy growing demand. See Van Mieghem [81] for a critical review of this literature. Lastly, our growth model is also similar to the new-product diffusion models under supply constraints (e.g., Ho et al. [41], and Kumar and Swaminathan [49]). We contribute to these two streams by studying the impact of financial constraints and bankruptcy risk on a firm’s growth strategy. We capture the firm’s growth prospects with a single parameter, which makes our model relatively more tractable.

2.3 Growth without Bankruptcy Risk

We analyze operational and financial decisions of a firm that faces random demand. The firm is managed by the owner who seeks to maximize the dis-

counted value of her firm's expected ending cash position in a T -period problem. In this section, the owner avoids making risky financial decisions to ensure that her company survives even in the worst demand realizations. This setting allows us to study the implications of a self-financing growth strategy.¹

It is conventional to index time by the number of periods left. For instance, period t implies that the firm has t more periods to go. The firm has $x_t \geq 0$ available as its starting capital. It orders Q_t units by paying cQ_t to its supplier, where c denotes the per unit cost. If $x_t > cQ_t$, the firm puts its excess cash, $x_t - cQ_t$, into the financial market to generate an interest revenue at a risk free rate r_f . If $x_t < cQ_t$, then the firm borrows $cQ_t - x_t$ to pay the supplier. It owes $(1+r_f)(cQ_t - x_t)$, i.e., loan plus interest, to the financial market at the end of period t . If the firm borrows, Q_t should be sufficiently small such that the firm can repay $(1 + r_f)(cQ_t - x_t)$ with probability one (i.e., regardless of the realized demand in period t). After the firm purchases Q_t units, a random demand D_t occurs. The firm generates a revenue of $p \min\{Q_t, D_t\} + s(Q_t - D_t)^+$, where p denotes the selling price, s denotes the salvage value of unsold units, and $x^+ = \max\{x, 0\}$. The firm salvages unsold units at the end of the period. Hence, it does not carry inventory from one period to the next. Carrying inventory makes the bankruptcy process and pricing risk more complicated, but does not change the main tradeoff between growth and bankruptcy risk.

The firm starts its operations with a starting cash position of $x_T \geq 0$. Its ending cash position in period t (i.e., starting cash position in period $t - 1$) can

¹In practice, a self-financing growth strategy refers to a growth strategy funded solely by retained earnings. In our model, we slightly modify this definition, and allow the firm to borrow without risk. In our model, borrowing without risk can be interpreted as borrowing from a supplier, i.e., trade credit.

be written as

$$x_{t-1} = (1 + r_f)(x_t - cQ_t) + p \min\{Q_t, D_t\} + s(Q_t - D_t)^+ \quad \text{for } t \in \{1, \dots, T\}.$$

In this formulation, $x_t - cQ_t$ represents the firm's excess cash or outstanding debt. $D_T = \xi_T$ where ξ_T is a random variable. For $t < T$, we assume that the firm's demand D_t has the following form:

$$D_t = \gamma \min\{Q_{t+1}, D_{t+1}\} + \xi_t. \quad (2.1)$$

Here $\gamma \geq 0$ is a scalar, which captures the firm's growth prospects. Following (2.1), demand in period t , D_t , is a function of the firm's growth prospects, the past demand realizations ξ_T, \dots, ξ_{t+1} and the past order quantities Q_T, \dots, Q_{t+1} . This setting allows us to capture path dependence, where a high order quantity in period t stochastically increases demand in the subsequent periods. In (2.1), the first component, $\gamma \min\{Q_{t+1}, D_{t+1}\}$, is a deterministic function sales in period $t + 1$. Let $y_t = \gamma \min\{Q_{t+1}, D_{t+1}\}$ and $q_t = Q_t - y_t$. Then, sales in period t can be written as

$$\begin{aligned} \min\{Q_t, D_t\} &= \min\{y_t + q_t, \gamma \min\{Q_{t+1}, D_{t+1}\} + \xi_t\} \\ &= \min\{y_t + q_t, y_t + \xi_t\} \\ &= y_t + \min\{q_t, \xi_t\}. \end{aligned} \quad (2.2)$$

Observe that sales in period t , $\min\{Q_t, D_t\}$, decomposes into two components. The first component, y_t , can be interpreted as the *baseline demand* that the firm builds in periods $T, \dots, t + 1$. This component is deterministic at the beginning of period t . The second component, $\min\{q_t, \xi_t\}$ is random. From an inventory perspective, y_t and q_t can be interpreted as orders placed to satisfy the deterministic and the stochastic components of demand, respectively. The stochastic

component of demand ξ_t is non-negative and follows a continuous probability distribution. $\xi_t \doteq \xi$ are i.i.d. across time. The pdf, cdf, complementary cdf (ccdf), and inverse ccdf of the demand distribution are denoted as f, F, \bar{F} , and \bar{F}^{-1} , respectively. Let $\beta = 1/(1 + r_f)$ denote the discount rate, and let $\bar{p} = \beta p$ and $\bar{s} = \beta s$. Using $Q_t = y_t + q_t$ and (2.2), we can write x_{t-1} as

$$\begin{aligned} x_{t-1} &= (1 + r_f)(x_t - cQ_t) + p \min\{Q_t, D_t\} + s(Q_t - D_t)^+ \\ &= (1 + r_f)[x_t + (\bar{p} - c)y_t + (\bar{p} - \bar{s}) \min\{q_t, \xi_t\} - (c - \bar{s})q_t] \end{aligned}$$

This formula allows us to write the firm's ending cash position as a function of its starting cash position x_t , baseline demand y_t , and a newsvendor type sales revenue formula $(\bar{p} - \bar{s}) \min\{q_t, \xi_t\} - (c - \bar{s})q_t$ with i.i.d. random demand $\xi_t \geq 0$. Let $y_T \geq 0$ denote the firm's starting baseline demand. Then the baseline demand at the end of period $t \in \{1, \dots, T\}$ can be written recursively as

$$\begin{aligned} y_{t-1} &= \gamma \min\{Q_t, D_t\} \\ &= \gamma(y_t + \min\{q_t, \xi_t\}), \end{aligned}$$

where the second line follows from (2.2).

Recall that the firm avoids making financially risky decisions. Also observe that the firm knows y_t before making its ordering decision. The firm has no bankruptcy risk if $x_{t-1} \geq 0$ even when $\xi_t = 0$. Setting $\xi_t = 0$ in (2.3) shows that the firm has no bankruptcy risk if

$$q_t \leq \frac{x_t + (\bar{p} - c)y_t}{c - \bar{s}}. \quad (2.3)$$

This constraint limits the order quantity, and ensures that the firm can borrow at a risk free rate because the firm repays loan plus interest in full with probability one.

The firm's objective is to choose its order quantities (q_T, \dots, q_1) to maximize the discounted value of its expected ending cash position. This problem can be formulated as a dynamic program. Let $\pi_t^C(q_t|x_t, y_t)$ denote the discounted expected terminal wealth when the firm orders q_t in period t and the optimal order quantity in the subsequent periods. Here, the superscript C denotes constrained growth. π_t is a function of the state variables x_t and y_t and the decision variable q_t . Similarly, let $V_t^C(x_t, y_t)$ denote the maximum discounted expected terminal wealth when the firm has t periods to go. In the last period, the firm seeks to maximize its discounted expected ending cash position subject to the cash flow constraint. That is,

$$\begin{aligned} V_1^C(x_1, y_1) \equiv \max \quad & \pi_1^C(q_1|x_1, y_1) \equiv \beta \int_0^\infty x_0 f(\xi_1) d\xi_1 \\ \text{s.t.} \quad & x_0 = (1 + r_f)[x_1 + (\bar{p} - c)y_1 + (\bar{p} - \bar{s}) \min\{q_1, \xi_1\} - (c - \bar{s})q_1] \\ & 0 \leq q_1 \leq \frac{x_1 + (\bar{p} - c)y_1}{c - \bar{s}}. \end{aligned}$$

For $t > 1$,

$$\begin{aligned} V_t^C(x_t, y_t) \equiv \max \quad & \pi_t^C(q_t|x_t, y_t) \equiv \beta \int_0^\infty V_{t-1}^C(x_{t-1}, y_{t-1}) f(\xi_t) d\xi_t \\ \text{s.t.} \quad & x_{t-1} = (1 + r_f)[x_t + (\bar{p} - c)y_t + (\bar{p} - \bar{s}) \min\{q_t, \xi_t\} - (c - \bar{s})q_t] \\ & y_{t-1} = \gamma(y_t + \min\{q_t, \xi_t\}) \\ & 0 \leq q_t \leq \frac{x_t + (\bar{p} - c)y_t}{c - \bar{s}}. \end{aligned} \tag{2.4}$$

The first two constraints track the state variables. The cash flow constraint (2.4) ensures that the firm avoids making risky investment decisions. We first show two technical results that will be useful to characterize the optimal ordering policy. Then we analyze the best case scenario in which the firm has sufficient cash in each period so that the cash flow constraints are never binding.

Lemma 2.3.1 For any period t , $V_t^C(x, y)$ increases in x for a fixed y , and increases in y for a fixed x .

Proof is omitted, and follows from the fact that both variables expand the constraint set.

Lemma 2.3.2 $\pi_t^C(q|x, y)$ is jointly concave in (x, y, q) , and $V_t^C(x, y)$ is jointly concave in (x, y) .

Proof: The objective function is the last period is

$$\begin{aligned}\pi_1^C(q|x, y) &= \int_0^\infty x_0 f(\xi_1) d\xi_1 \\ &= x + (\bar{p} - c)y + (\bar{p} - \bar{s}) \int_0^q \bar{F}(\xi_1) d\xi_1 - (c - \bar{s})q,\end{aligned}$$

which is jointly concave in (x, y, q) . Hence, $V_1^C(x, y)$ is jointly concave in (x, y) because the constraint set is convex. Next we show that $V_t^C(x, y)$ is jointly concave in (x, y) for $t \geq 2$. Suppose $V_{t-1}^C(x, y)$ is jointly concave in (x, y) by induction. We first show that $V_{t-1}^C((1 + r_f)[x + (\bar{p} - c)y + (\bar{p} - \bar{s}) \min\{q, \xi_t\} - (c - \bar{s})q], \gamma(v + \min\{q, \xi_t\}))$ is jointly concave in (x, y, q) . Let $\lambda \in [0, 1]$ and $\bar{\lambda} = 1 - \lambda$. For any (x, y, q) and $(\bar{x}, \bar{y}, \bar{q})$, let $\bar{x} = \lambda x + \bar{\lambda} \bar{x}$, $\bar{y} = \lambda y + \bar{\lambda} \bar{y}$, and $\bar{q} = \lambda q + \bar{\lambda} \bar{q}$. Furthermore let $\hat{q} = \lambda \min\{q, \xi_t\} + \bar{\lambda} \min\{\bar{q}, \xi_t\}$. Then

$$\begin{aligned}& V_{t-1}^C \left((1 + r_f)[\bar{x} + (\bar{p} - \bar{s})\bar{y} + (\bar{p} - \bar{s}) \min\{\bar{q}, \xi_t\} - (c - \bar{s})\bar{q}], \gamma(\bar{y} + \min\{\bar{q}, \xi_t\}) \right) \\ & \geq V_{t-1}^C \left((1 + r_f)[\bar{x} + (\bar{p} - \bar{s})\bar{y} + (\bar{p} - \bar{s})\hat{q} - (c - \bar{s})\bar{q}], \gamma(\bar{y} + \min\{\bar{q}, \xi_t\}) \right) \\ & \geq V_{t-1}^C \left((1 + r_f)(\bar{x} - c(\bar{q} + \bar{y})) + p\bar{y} + (p - s)\hat{q} + s\bar{q}, \gamma(\bar{y} + \hat{q}) \right) \\ & \geq \lambda V_{t-1}^C \left((1 + r_f)[x + (\bar{p} - \bar{s})y + (\bar{p} - \bar{s}) \min\{q, \xi_t\} + sq], \gamma(y + \min\{q, \xi_t\}) \right) \\ & \quad + \bar{\lambda} V_{t-1}^C \left((1 + r_f)[\bar{x} + (\bar{p} - \bar{s})\bar{y} + (\bar{p} - \bar{s}) \min\{\bar{q}, \xi_t\} + s\bar{q}], \gamma(\bar{y} + \min\{\bar{q}, \xi_t\}) \right).\end{aligned}$$

The first inequality follows because $\min\{\bar{q}, \xi_t\} \geq \hat{q}$ and $V_{t-1}^C(x, y)$ is an increasing function of x . The second inequality follows because $V_{t-1}^C(x, y)$ is an increasing function of y . The last inequality follows because $V_{t-1}^C(x, y)$ is jointly concave in (x, y) . Hence, $V_{t-1}^C((1 + r_f)[x + (\bar{p} - c)y + (\bar{p} - \bar{s}) \min\{q, \xi_t\} - (c - \bar{s})q], \gamma(y + \min\{q, \xi_t\}))$ is jointly concave in (x, y, q) . Consequently, $\pi_t^C(q|x, y) = \beta E[V_{t-1}^C((1 + r_f)[x + (\bar{p} - c)y + (\bar{p} - \bar{s}) \min\{q, \xi_t\} - (c - \bar{s})q], \gamma(y + \min\{q, \xi_t\}))]$ is also jointly concave in (x, y, q) . Then

$$V_t^C(x, y) = \max_{0 \leq q \leq \frac{x + (\bar{p} - c)y}{c - \bar{s}}} \pi_t^C(q|x, y)$$

is jointly concave in (x, y) since the constraint set is convex. \square

2.3.1 Unconstrained Growth

Suppose (x_t, y_t) are sufficiently high such that the cash flow constraint does not bind at the optimal solution. The optimal solution of this problem reveals the firm's unconstrained growth trajectory. The next proposition formalizes this argument. This policy will serve as a benchmark for our subsequent analysis because it allows the firm to achieve growth without running into cash flow problems.

Proposition 2.3.1 *Let*

$$N_t(q) \equiv \left(c - \bar{s} + (\bar{p} - c) \sum_{i=0}^{t-1} \bar{\gamma}^i \right) \int_0^q \bar{F}(\xi_t) d\xi_t - (c - \bar{s})q,$$

where $\bar{\gamma} = \beta\gamma$. Furthermore, let

$$q_t^U \equiv \bar{F}^{-1} \left(\frac{c - \bar{s}}{c - \bar{s} + (\bar{p} - c) \sum_{i=0}^{t-1} \bar{\gamma}^i} \right). \quad (2.5)$$

If

$$x_t + (\bar{p} - c) \sum_{i=0}^{t-1} \bar{\gamma}^i y_t \geq (c - \bar{s}) \sum_{i=1}^t \beta^{t-i} q_i^U \quad (2.6)$$

then the optimal order quantity that is used to satisfy the random component of demand in period t is equal to q_t^U , and the corresponding value function is

$$V_t^C(x_t, y_t) \equiv x_t + (\bar{p} - c) \sum_{i=0}^{t-1} \tilde{\gamma}^i y_t + \sum_{l=1}^t \beta^{t-l} N_l(q_l^U) \quad \text{for } t \in \{1, \dots, T\}.$$

Proof: Proof is by induction. In the last period, the firm solves

$$\begin{aligned} V_1^C(x_1, y_1) = & \max_{q_1} x_1 + (\bar{p} - c)y_1 + (\bar{p} - \bar{s}) \int_0^{q_1} \bar{F}(\xi_1) d\xi_1 - (c - \bar{s})q_1 \\ \text{s.t. } & 0 \leq q_1 \leq \frac{x_1 + (\bar{p} - c)y_1}{c - \bar{s}}. \end{aligned}$$

$q_1^U = \bar{F}^{-1}\left(\frac{c - \bar{s}}{\bar{p} - \bar{s}}\right)$ maximizes the objective function in the absence of the constraint. Hence, if $\frac{x_1 + (\bar{p} - c)y_1}{c - \bar{s}} \geq q_1^U$, we have $V_1^C(x_1, y_1) = x_1 + (\bar{p} - c)y_1 + N_1(q_1^U)$. Suppose the formula holds in period $t - 1$. In period t , suppose $x_t + (\bar{p} - c) \sum_{i=0}^{t-1} \tilde{\gamma}^i y_t \geq (c - \bar{s}) \sum_{i=1}^t \beta^{t-i} q_i^U$. If the firm orders q_t^U , we have

$$\begin{aligned} x_{t-1} + (\bar{p} - c) \sum_{i=0}^{t-2} \tilde{\gamma}^i y_{t-1} &= (1 + r_f)[x_t + (\bar{p} - c)y_t + (\bar{p} - \bar{s}) \min\{q_t^U, \xi_t\} - (c - \bar{s})q_t^U] \\ &\quad + (\bar{p} - c) \sum_{i=0}^{t-2} \tilde{\gamma}^i [\gamma(y_t + \min\{q_t^U, \xi_t\})] \\ &\geq (1 + r_f)[x_t + (\bar{p} - c)y_t - (c - \bar{s})q_t^U] + (\bar{p} - c) \sum_{i=0}^{t-2} \tilde{\gamma}^i \gamma y_t \\ &= (1 + r_f) \left[x_t + (\bar{p} - c) \sum_{i=0}^{t-1} \tilde{\gamma}^i y_t - (c - \bar{s})q_t^U \right] \\ &\geq (1 + r_f) \left[(c - \bar{s}) \sum_{i=1}^t \beta^{t-i} q_i^U - (c - \bar{s})q_t^U \right] \\ &= (c - \bar{s}) \sum_{i=1}^{t-1} \beta^{t-i-1} q_i^U. \end{aligned}$$

The first inequality follows because the worst demand realization is equal to zero. The second inequality follows from (2.6). Hence, if (2.6) holds in period t and the firm orders q_t^U then we know that (2.6) will also hold in period $t - 1$. By

induction,

$$\begin{aligned}\pi_t^C(q_t^U|x_t, y_t) &= \beta \int_0^\infty V_{t-1}^C(x_{t-1}, y_{t-1}) f(\xi_t) d\xi_t \\ &= \beta \int_0^\infty \left(x_{t-1} + (\bar{p} - c) \sum_{i=0}^{t-2} \tilde{\gamma}^i y_{t-1} + \sum_{l=1}^{t-1} \beta^{t-l-1} N_l(q_l^U) \right) f(\xi_t) d\xi_t.\end{aligned}$$

Using the definitions of x_{t-1} and y_{t-1} , we can rewrite $\pi_t^C(q_t^U|x_t, y_t)$ as

$$\begin{aligned}\pi_t^C(q_t^U|x_t, y_t) &= \beta \int_0^\infty \left(x_{t-1} + (\bar{p} - c) \sum_{i=0}^{t-2} \tilde{\gamma}^i y_{t-1} + \sum_{l=1}^{t-1} \beta^{t-l-1} N_l(q_l^U) \right) f(\xi_t) d\xi_t \\ &= x_t + (\bar{p} - c) \sum_{i=0}^{t-1} \tilde{\gamma}^i y_t + \left(c - \bar{s} + (\bar{p} - c) \sum_{i=0}^{t-1} \tilde{\gamma}^i \right) \int_0^{q_t^U} \bar{F}(\xi_t) d\xi_t \\ &\quad - (c - \bar{s}) q_t^U + \sum_{l=1}^{t-1} \beta^{t-l} N_l(q_l^U) \\ &= x_t + (\bar{p} - c) \sum_{i=0}^{t-1} \tilde{\gamma}^i y_t + \sum_{l=1}^t \beta^{t-l} N_l(q_l^U).\end{aligned}$$

Furthermore, the second line indicates that $\frac{\partial \pi_t^C}{\partial q_t} \big|_{q_t=q_t^U} = 0$. Hence, q_t^U is the optimal order quantity because π_t is concave in q by Lemma 2.3.2. \square

Observe that the objective function decomposes into T separate single-period inventory problems when the cash flow constraints are non-binding. If $\gamma = 0$ then the optimal order quantity in period t is equal to $\bar{F}\left(\frac{c-\bar{s}}{\bar{p}-\bar{s}}\right)$, which is the classical newsvendor solution. However, the order placed in period t has an impact on future demand when $\gamma > 0$. This impact leads to the *modified newsvendor solution*, q_t^U , which is greater than the classical newsvendor solution. That is, the firm orders more than the classical newsvendor solution when the order quantity has a positive impact on future demand.

The firm's total order quantity in period t is

$$Q_t^U = \begin{cases} q_T^U & \text{if } t = T, \\ \gamma \min\{Q_{t+1}^U, D_{t+1}\} + q_t^U & \text{if } t < T. \end{cases} \quad (2.7)$$

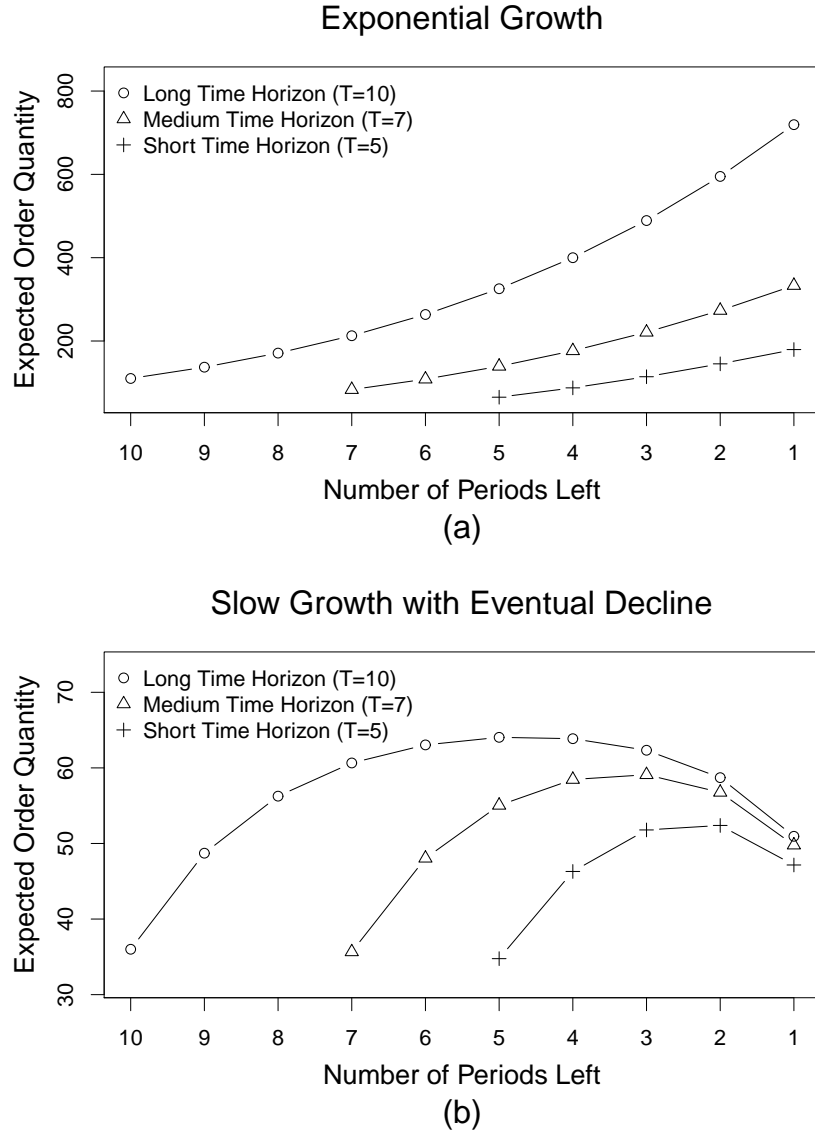
Q_t^U defines the firm's unconstrained growth trajectory. It can also be interpreted as the self-financing growth trajectory because the firm either does not borrow or borrows a minimal amount that is always repaid.

At the beginning of the time horizon, Q_t^U values are random variables for $t \in \{1, \dots, T-1\}$ because they depend on past demand realizations, ξ_T, \dots, ξ_{t+1} . For planning purposes, the firm can compute various performance measures related to Q_t^U . For example it can compute the expected order quantities $E_T[Q_t^U]$, where the subscript T denotes the firm's information set at the beginning of the time horizon. Other moments of Q_t^U can also be computed to assess the uncertainty associated with the firm's unconstrained growth trajectory. Figure 2.2 shows the expected order quantities as a function of the length of the time horizon T and the growth coefficient γ . T determines the magnitude of the firm's growth prospects. For example, in Figure 2.2(a), $T = 10$ allows the firm to have significantly higher order quantities compared to $T = 5$. γ determines the nature of the firm's growth trajectory. For example, $\gamma > 1$ implies monotone growth, whereas $\gamma < 1$ captures slow growth with eventual decline.

Despite its simplicity, our demand model can capture a wide variety of growth trajectories. If one is interested in modeling more complicated growth trajectories (e.g., a convex increasing expected order quantity followed by an eventual decline), s/he can do so by making the growth coefficients and/or demand variables time specific. For our purposes, a combination of a time invariant γ value and identically distributed demand shocks are sufficient. Hence, we confine our attention to this scenario in the rest the chapter.

Proposition 2.3.1 also defines the necessary conditions to achieve the maximum growth with self-financing. Namely, (2.6) implies that the firm should

Figure 2.2: The expected order quantity $E_T[Q_t^U]$ for various scenarios. ξ_t are i.i.d. log-normal with $f(\xi_t) = \frac{1}{\xi \sqrt{2\pi\sigma^2}} \exp\left(-\frac{(\ln \xi - \mu)^2}{2\sigma^2}\right)$, where $E[\xi] = 33.12 = \sqrt{\text{Var}[\xi]} = 33.12$. $p = 2, c = 1, s = 0.2, \beta = 0.95$. In Figure (a), we compute the expected order quantities for three different problems with $T = 5, 7, 10$. The growth coefficient $\gamma = 1.2$. Figure (b) is identical to (a) except $\gamma = 0.6$.



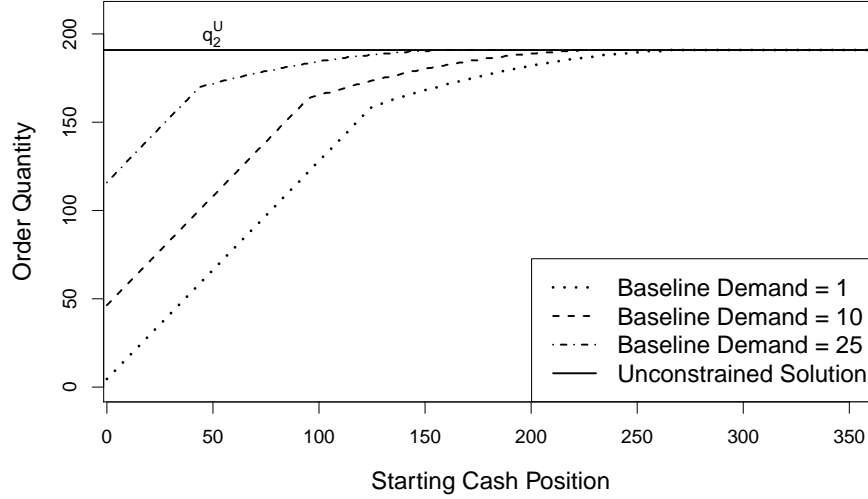
have a high starting capital and/or a strong baseline demand to be able to afford ordering the modified newsvendor quantities. Starting with a large (x_T, y_T) combination is one way to ensure unconstrained growth. However, a firm that starts its operations with cash flow constraints might also reach the unconstrained regime through a combination of the optimal order quantities and a sequence of high demand realizations. In the next subsection, we characterize the optimal ordering policy under the presence of cash flow constraints.

2.3.2 Constrained Growth

Let $q_t^C(x, y)$ denote the optimal order quantity of the constrained problem. It is relatively easy to compute q_t^C because $\pi_t^C(q|x, y)$ is concave in q . Figure 2.3 shows the optimal order quantity as a function of the firm's starting cash position and baseline demand. For any fixed baseline demand value, the optimal order quantity increases in the starting cash position. Similarly, it increases in the baseline demand for a given starting cash position. Observe that the optimal order quantity increases linearly for small values of x and y . That is, the credit limit is binding, and $q_t^C(x, y) = \frac{x+(\bar{p}-c)y}{c-\bar{s}}$ when x and y are relatively small. As x or y becomes large, the cash flow constraint becomes non-binding. That is, the optimal order quantity satisfies the first order condition, $\frac{\partial \pi_t^C}{\partial q} = 0$. Although the optimal order quantity equals the unconstrained solution, q_t^U , when (2.6) holds, there exists (x, y) values such that the constraint is non-binding, and the optimal solution is less than q_t^U . In those instances, the firm prefers to conserve cash rather than maxing out its credit limit to order q_t^U .

Another implication of Figure 2.3 is that a cash constrained firm with a very

Figure 2.3: The cash constrained optimal order quantity in period 2, q_2^C , as a function of x_2 and y_2 . $T = 2$ and $\xi_{i \in \{1,2\}}$ is exponential with mean 100. $p = 5$, $c = 1$, $s = 0.2$, $r_f = 0.05$, and $\gamma = 0.25$. The unconstrained order quantity q_2^U is equal to 191.01. We vary the starting cash position x_2 between 0 and 360 for three baseline demand values; $y_2 = 1$, $y_2 = 10$, and $y_2 = 25$. The optimal order quantity is equal to the cash flow constraint $\frac{x_2 + (\bar{p} - c)y_2}{c - \bar{s}}$ for small values of x_2 and y_2 . For relatively large values of x_2 and y_2 , the credit constraint is not binding, and the optimal order quantity satisfies the first order condition $\frac{\partial \pi_2^C}{\partial q_2} = 0$.



low (x_t, y_t) value enters a vicious cycle because it can only order a small amount, which leads to tiny profits. As a result, (x_{t-1}, y_{t-1}) is also low, and the firm grows at a very slow pace. One alternative for the firm to break such a vicious cycle is to borrow a relatively large amount (i.e., order more than $\frac{x + (\bar{p} - c)y}{c - \bar{s}}$), which introduces bankruptcy risk. We analyze risky borrowing in the next section.

2.4 Growth with Bankruptcy Risk

In the previous section, we imposed a constraint on the firm's order quantity to ensure that its ending cash position is non-negative even when the realized demand equals zero. In this section, we remove this constraint, which intro-

duces bankruptcy risk. As a consequence, the risk neutral lending market sets the lending interest rate to price the firm's bankruptcy risk. Let α_t denotes the lending interest rate. Then the firm's ending cash position in period t can be written as

$$x_{t-1} = (1+r_f)[x_t - c(y_t + q_t)]^+ + py_t + (p-s) \min\{q_t, \xi_t\} + sq_t - (1+\alpha_t)[c(y_t + q_t) - x_t]^+. \quad (2.8)$$

This formula implies that if $c(y_t + q_t) \leq x_t$ then the firm does not borrow, and puts its excess cash into the financial market to generate an interest revenue at the risk free rate r_f . If $c(y_t + q_t) > x_t$ then the firm borrows, and the risk neutral financial market sets the lending interest rate α_t to earn the risk free rate r_f in expectation. Let $z_t(q_t, \alpha_t | x_t, y_t)$ denote the threshold demand value below which the firm's ending cash position is negative. Setting $\min\{q_t, \xi_t\} = z_t(q_t, \alpha_t | x_t, y_t)$ in (2.8) and rearranging terms show that

$$z_t(q_t, \alpha_t | x_t, y_t) = \left(\frac{[(1 + \alpha_t)c - s]q_t - [p - (1 + \alpha_t)c]y_t - (1 + \alpha_t)x_t}{p - s} \right)^+. \quad (2.9)$$

If the firm borrows and the realized demand exceeds z_t then the bank receives the loan plus interest $(1 + \alpha_t)[c(q_t + y_t) - x_t]$ in full. If the realized demand is less than z_t then the firm does not have sufficient cash to repay the loan plus interest. As a result, the bank only receives $py_t + (p - s)\xi_t + sq_t$. The bank determines the lending interest rate using the following formula.

Lemma 2.4.1 *If $q_t \leq \frac{x_t + (\bar{p} - c)y_t}{c - \bar{s}}$, then the firm has no bankruptcy risk (NBR) in period t , and $\alpha_t = r_f$. If $q_t > \frac{x_t + (\bar{p} - c)y_t}{c - \bar{s}}$, then the firm borrows with risk (BWR), and α_t is set such that*

$$\int_0^{z_t(q_t, \alpha_t | x_t, y_t)} \bar{F}(\xi_t) d\xi_t = \frac{(c - \bar{s})q_t - (\bar{p} - c)y_t - x_t}{\bar{p} - \bar{s}}. \quad (2.10)$$

Proof: α_t must be equal to r_f when the firm has no bankruptcy risk because the firm repays the loan plus interest in full. As we discussed in the previous

section, the firm has no bankruptcy risk if $q_t \leq \frac{x_t + (\bar{p} - c)y_t}{c - \bar{s}}$. If the firm orders $q_t > \frac{x_t + (\bar{p} - c)y_t}{c - \bar{s}}$ then the lender receives

$$\min\{py_t + (p - s)\xi_t + sq_t, (1 + \alpha_t)[c(y_t + q_t) - x_t]\}.$$

Hence, the bank's expected return is

$$\frac{E[\min\{py_t + (p - s)\xi_t + sq_t, (1 + \alpha_t)[c(y_t + q_t) - x_t]\}]}{c(q_t + y_t) - x_t} - 1.$$

Setting it equal to r_f and rearranging terms give (2.10). \square

We shall drop the arguments of $z_t(q_t, \alpha_t | x_t, y_t)$ for notational convenience. Observe that if the firm borrows with risk then

$$\begin{aligned} x_{t-1} &= (1 + \alpha_t)x_t + [p - (1 + \alpha_t)c]y_t + (p - s) \min\{q_t, \xi_t\} - [(1 + \alpha_t)c - s]q_t \\ &= (p - s) \left(\min\{q_t, \xi_t\} - \frac{[(1 + \alpha_t)c - s]q_t - [p - (1 + \alpha_t)c]y_t - (1 + \alpha_t)x_t}{p - s} \right) \\ &= (p - s) (\min\{q_t, \xi_t\} - z_t). \end{aligned}$$

The last line follows from (2.9) because $z_t > 0$ when the firm borrows with risk.

Hence, the firm's ending cash position can be written as

$$x_{t-1} = \begin{cases} (1 + r_f)[x_t + (\bar{p} - c)y_t + (\bar{p} - \bar{s}) \min\{q_t, \xi_t\} - (c - \bar{s})q_t] & \text{if } q_t \leq \frac{x_t + (\bar{p} - c)y_t}{c - \bar{s}}, \\ (p - s) (\min\{q_t, \xi_t\} - z_t) & \text{if } q_t > \frac{x_t + (\bar{p} - c)y_t}{c - \bar{s}}. \end{cases}$$

Observe that x_{t-1} above is identical to (2.3) when the firm has no bankruptcy risk. When the firm borrows with risk, it earns (loses) $p - s$ for every unit of demand that is above (below) the bankruptcy threshold z_t .

What happens if the ending cash position falls below zero? A negative ending cash position implies that the firm cannot meet its debt obligations. In practice, a firm that cannot meet its debt obligations can file for either a Chapter 7

or a Chapter 11 bankruptcy. Under Chapter 7 bankruptcy, it liquidates its assets and stops operating. Under Chapter 11 bankruptcy, it reorganizes itself and continues to operate. Of course, bankruptcy in practice is a complicated procedure with many stake holders and legal claimants. In the rest of this section, we will explore two extreme forms of bankruptcy in terms of their financial implications for the owner. First, we will analyze a Chapter 11 bankruptcy in which the firm restructures its operations without losing its customer base. In our model, the firm's remaining debt to the bank is erased, and it continues to operate with zero starting cash in the following period. The bank takes this possibility into account while setting the lending interest rate using (2.10). Hence, it is not an unfair scenario from the bank's perspective. Then we will analyze a Chapter 7 bankruptcy in which the firm stops its operations in the first period in which its ending cash position falls below zero. In practice, debt holders and sometimes equity holders receive some additional payments due to the liquidation of the firm's assets or the sales of the firm's intangible assets (in our model the firm's baseline demand can be interpreted as its market share). However, we omit these additional benefits for simplicity. By studying these two extremes, we analyze the best and the worst case scenarios for the owner in case of a bankruptcy.

2.4.1 Reorganization Bankruptcy

In this section, we analyze a scenario in which the firm can continue to operate after declaring bankruptcy. In this scenario, the firm declares bankruptcy when its cash position falls below zero. After declaring bankruptcy, the firm's outstanding debt is erased, and it is allowed to have a fresh restart without los-

ing its baseline demand. We show the optimality of the modified newsvendor policy we presented in Proposition 2.3.1. That is, the firm solves T separate modified newsvendor problems. This result has the following interpretation: if a bankrupt firm is allowed to resume its operations without any penalties, it will make its operational decisions as if it were solving an unconstrained growth problem. We first present the dynamic programming formulation of this problem. In the last period, the firm solves

$$\begin{aligned}
V_1^R(x_1, y_1) &\equiv \max_{q_1 \geq 0} & \pi_1^R(q_1|x_1, y_1) &\equiv \beta \int_0^\infty \max\{x_0, 0\} f(\xi_1) d\xi_1 \\
\text{s.t.} & & x_0 &= (1 + r_f)[x_1^+ - c(y_1 + q_1)]^+ + py_1 + (p - s) \min\{q_1, \xi_1\} \\
& & &+ sq_1 - (1 + \alpha_1)[c(y_1 + q_1) - x_1^+]^+ \\
& & \int_0^{z_1} \bar{F}(\xi_1) d\xi_1 &\geq \frac{(c - \bar{s})q_1 - (\bar{p} - c)y_1 - x_1^+}{\bar{p} - \bar{s}} \\
& & \alpha_1 &\geq r_f \\
& & z_1 &= \left(\frac{[(1 + \alpha_1)c - s]q_1 - [p - (1 + \alpha_1)c]y_1 - (1 + \alpha_1)x_1^+}{p - s} \right)^+.
\end{aligned}$$

In periods $\{T, \dots, 2\}$, the firm solves

$$\begin{aligned}
V_t^R(x_t, y_t) &= \max_{q_t \geq 0} & \pi_t^R(q_t|x_t, y_t) &\equiv \beta \int_0^\infty V_{t-1}^R(x_{t-1}, y_{t-1}) f(\xi_t) d\xi_t \\
\text{s.t.} & & x_{t-1} &= (1 + r_f)[x_t^+ - c(q_t + y_t)]^+ + py_t + (p - s) \min\{q_t, \xi_t\} \\
& & &+ sq_t - (1 + \alpha_t)[c(q_t + y_t) - x_t^+]^+ \\
& & y_{t-1} &= \gamma(y_t + \min\{q_t, \xi_t\}) \\
& & \int_0^{z_t} \bar{F}(\xi_t) d\xi_t &\geq \frac{(c - \bar{s})q_t + (\bar{p} - c)y_t - x_t^+}{\bar{p} - \bar{s}} \tag{2.11} \\
& & \alpha_t &\geq r_f \tag{2.12} \\
& & z_t &= \left(\frac{[(1 + \alpha_t)c - s]q_t - [p - (1 + \alpha_t)c]y_t - (1 + \alpha_t)x_t^+}{p - s} \right)^+.
\end{aligned}$$

Observe that the state variable x_t appears as x_t^+ in the optimization model because if the firm's starting cash position is negative then it declares bankruptcy

and restarts its operations with zero cash. If the firm has bankruptcy risk, (2.11) becomes binding. Otherwise, (2.12) is binding, and the lending interest rate equals r_f . The following result will be useful to characterize the optimal policy.

Lemma 2.4.2 *If the firm borrows with risk in period t , then*

$$\begin{aligned} E[\max\{x_{t-1}, 0\}] &= (p - s) \int_{z_t}^{\infty} (\min\{q_t, \xi_t\} - z_t) f(\xi_t) d\xi_t \\ &= (1 + r_f) \left[x_t^+ + (\bar{p} - c)y_t + (\bar{p} - \bar{s}) \int_0^{q_t} \bar{F}(\xi_t) d\xi_t - (c - \bar{s})q_t \right]. \end{aligned}$$

Proof: Recall from our definition of reorganization bankruptcy that if x_t is negative then the firm starts period t with zero cash. Hence, the firm's starting cash position is x_t^+ . If $q_t > \frac{x_t^+ + (\bar{p} - c)y_t}{c - \bar{s}}$ (i.e., if the firm borrows with risk) then its ending cash position is non-negative only when $\xi_t \geq z_t$. Hence,

$$E[\max\{x_{t-1}, 0\}] = (p - s) \int_{z_t}^{\infty} (\min\{q_t, \xi_t\} - z_t) f(\xi_t) d\xi_t.$$

Borrowing with risk implies that $z_t(q_t, \alpha_t | x_t^+, y_t) > 0$. As a result, α_t is determined by (2.10). Then

$$\begin{aligned} E[x_{t-1}^+] &= (p - s) \int_{z_t}^{\infty} (\min\{q_t, \xi_t\} - z_t) f(\xi_t) d\xi_t \\ &= (p - s) \int_0^{\infty} (\min\{q_t, \xi_t\} - \min\{z_t, \xi_t\}) f(\xi_t) d\xi_t \\ &= (p - s) \left(\int_0^{q_t} \bar{F}(\xi_t) d\xi_t - \int_0^{z_t} \bar{F}(\xi_t) d\xi_t \right) \\ &= (p - s) \left(\int_0^{q_t} \bar{F}_t(\xi_t) d\xi_t - \frac{(c - \bar{s})q_t - (\bar{p} - c)y_t - x_t^+}{\bar{p} - \bar{s}} \right) \\ &= (1 + r_f) \left[x_t^+ + (\bar{p} - c)y_t + (\bar{p} - \bar{s}) \int_0^{q_t} \bar{F}_t(\xi_t) d\xi_t - (c - \bar{s})q_t \right]. \end{aligned}$$

The penultimate line follows from (2.10). □

The next proposition shows that the optimal order quantity that is used to satisfy the random component of demand is equal to q_t^U .

Proposition 2.4.1 *The optimal order quantity in period t , $q_t^R(x_t, y_t)$, is equal to $q_t^U = \bar{F}^{-1}\left(\frac{c-\bar{s}}{(\bar{p}-\bar{s})+(\bar{p}-c)\sum_{i=1}^{t-1}\bar{\gamma}^i}\right)$. Furthermore,*

$$V_t^R(x_t, y_t) = x_t^+ + (\bar{p} - c) \sum_{j=0}^{t-1} \bar{\gamma}^j y_t + \sum_{l=1}^t \beta^{t-l} N_l(q_l^U) \quad \text{for } t \in \{1, \dots, T\} \text{ and } x_t \in \mathbb{R}.$$

Proof: In the last period, if the firm chooses an order quantity that requires borrowing with risk (BWR), the objective function becomes

$$\begin{aligned} \pi_1^R(q_1|x_1, y_1, BWR) &= \beta E[\max\{x_0, 0\}] \\ &= x_1^+ + (\bar{p} - c)y_1 + (\bar{p} - \bar{s}) \int_0^{q_1} \bar{F}(\xi_1) d\xi_1 - (c - \bar{s})q_1. \end{aligned} \quad (2.13)$$

The second line follows from Lemma 2.4.2. If the firm chooses an order quantity that does not require borrowing with risk (NBR), the objective function is

$$\begin{aligned} \pi_1^R(q_1|x_1, y_1, NBR) &= E[\max\{x_0, 0\}] \\ &= E[(1 + r_f)[x_1^+ - c(y_1 + q_1)] + py_1 + p \min\{q_1, \xi_1\} + sq_1] \\ &= \pi_1^R(q_1|x_1, y_1, BWR), \end{aligned}$$

where the second line follows from setting $\alpha_1 = r_f$ in (2.11). Therefore, the objective functions under (BWR) and (NBR) are identical. $q^R(x_1, y_1) = q_1^U = \bar{F}^{-1}\left(\frac{c-\bar{s}}{\bar{p}-\bar{s}}\right)$ maximizes this objective function. Hence,

$$V_1^R(x_1, y_1) = x_1^+ + (\bar{p} - c)y_1 + N_1(q_1^U).$$

Now suppose the value function formula holds in period $t-1$. Further, suppose the firm chooses an order quantity that requires borrowing with risk. Then

$$\begin{aligned} \pi_t^R(q_t|x_t, y_t, BWR) &= \pi_t^R(q_t|x_t^+, y_t, BWR) \\ &= \beta \int_{z_t}^{\infty} V_{t-1}^R(x_{t-1}, y_{t-1}) f(\xi_t) d\xi_t + \beta \int_0^{z_t} V_{t-1}^R(0, y_{t-1}) f(\xi_t) d\xi_t \\ &= \beta \int_{z_t}^{\infty} \left(x_{t-1} + (\bar{p} - c) \sum_{j=0}^{t-2} \bar{\gamma}^j y_{t-1} + \sum_{l=1}^{t-1} \beta^{t-l-1} N_l(q_l^U) \right) f(\xi_t) d\xi_t \end{aligned}$$

$$\begin{aligned}
& +\beta \int_0^{z_t} \left((\bar{p} - c) \sum_{j=0}^{t-2} \bar{\gamma}^j y_{t-1} + \sum_{l=1}^{t-1} \beta^{t-l-1} N_l(q_l^U) \right) f(\xi_t) d\xi_t \\
= & \beta \int_{z_t}^{\infty} ((p - s)(\min\{q_t, \xi_t\} - z_t)) f(\xi_t) d\xi_t \\
& +\beta(\bar{p} - c) \sum_{j=0}^{t-2} \bar{\gamma}^j \int_0^{\infty} (\gamma(y_t + \min\{q_t, \xi_t\})) f(\xi_t) d\xi_t \\
& + \sum_{l=1}^{t-1} \beta^{t-l} N_l(q_l^U).
\end{aligned}$$

The fourth equality follows from Lemma (2.4.2). Rearranging terms gives

$$\begin{aligned}
\pi_t^R(q_t|x_t, y_t, BWR) &= x_t^+ + (\bar{p} - c)y_t + (\bar{p} - \bar{s}) \int_0^{q_t} \bar{F}(\xi_t) d\xi_t - (c - \bar{s})q_t \\
&+ (\bar{p} - c) \sum_{j=1}^{t-1} \bar{\gamma}^j \left(y_t + \int_0^{q_t} \bar{F}(\xi_t) d\xi_t \right) + \sum_{l=1}^{t-1} \beta^{t-l} N_l(q_l^U) \\
= & x_t^+ + (\bar{p} - c) \sum_{j=0}^{t-1} \bar{\gamma}^j y_t + \left(\bar{p} - \bar{s} + (\bar{p} - c) \sum_{j=1}^{t-1} \bar{\gamma}^j \right) \int_0^{q_t} \bar{F}(\xi_t) d\xi_t \\
& - (c - \bar{s})q_t + \sum_{l=1}^{t-1} \beta^{t-l} N_l(q_l^U) \\
= & x_t^+ + (\bar{p} - c) \sum_{j=0}^{t-1} \bar{\gamma}^j y_t + N_t(q_t) + \sum_{l=1}^{t-1} \beta^{t-l} N_l(q_l^U).
\end{aligned}$$

Following the same steps to derive the objective function under no borrowing with risk leads to the same expression. Hence,

$$\pi_t^R(q_t|x_t, y_t) = x_t^+ + (\bar{p} - c) \sum_{j=0}^{t-1} \bar{\gamma}^j y_t + N_t(q_t) + \sum_{l=1}^{t-1} \beta^{t-l} N_l(q_l^U).$$

q_t^U maximizes $\pi_t^R(q_t|x_t, y_t)$, regardless of how the procurement of q_t^U is financed.

As a result,

$$V_t^R(x_t, y_t) = x_t^+ + (\bar{p} - c) \sum_{j=0}^{t-1} \bar{\gamma}^j y_t + \sum_{l=1}^t \beta^{t-l} N_l(q_l^U)$$

for all x_t and $t \in \{1, \dots, T\}$. □

Intuitively, if the firm borrows from a risk neutral lending market under re-organization bankruptcy, its bankruptcy risk in each period measured by the,

probability of bankruptcy $F(z_t(q_t^U, \alpha_t|x_t^+, y_t))$, is appropriately priced. Combining this result with a reorganization bankruptcy setting that erases the firm's debt obligations in case of a bankruptcy decomposes the firm's objective function into T separate problems. As a result of this decomposition, the modified newsvendor solution we derive under the unconstrained growth scenario, q_t^U , becomes optimal.

This result shows that a well functioning financial system (i.e., a risk neutral lending market that drives the lender's expected profits to the risk free rate) and a reorganization bankruptcy mechanism that protects the firm in the event of a bankruptcy by erasing its debt without damaging its baseline demand enable the firm to reach its maximum growth potential. In reality, financial markets are not perfectly competitive and a reorganization bankruptcy involves costs, so $V_T^R(x_T, y_T)$ can be interpreted as the most optimistic estimate of the owner's expected ending cash position.

Let $\mathbb{P}_t(q_t^U|x, y)$ denote the probability of declaring bankruptcy at the end of period t under reorganization bankruptcy. The optimality of the modified newsvendor solution allows us to capture the tradeoff between growth and bankruptcy risk.

Lemma 2.4.3 *For a given (x, y) , $\mathbb{P}_t(q_t^U|x, y)$ increases in the firm's growth prospects measured by γ .*

Proof: The firm orders q_t^U in period t . If $q_t^U \leq \frac{x^+ + (\bar{p}-c)y}{c-\bar{s}}$ then the $x_{t-1} \geq 0$ with probability one. Hence, the firm has no bankruptcy risk. If $q_t^U > \frac{x^+ + (\bar{p}-c)y}{c-\bar{s}}$ then the firm borrows with risk, and (2.10) holds. Differentiating (2.10) with respect

to q_t gives

$$\left(\frac{\partial z_t}{\partial q_t} + \left(\frac{\partial z_t}{\partial \alpha_t} \right) \frac{\partial \alpha_t}{\partial q_t} \right) \bar{F}(z_t) = \frac{c - \bar{s}}{\bar{p} - \bar{s}}. \quad (2.14)$$

Furthermore, $\mathbb{P}_t(q_t^U | x, y) = \Pr(\xi_t < z_t(q_t^U, \alpha_t | x, y)) = F(z_t)$. Therefore,

$$\frac{\partial \mathbb{P}_t}{\partial \gamma} = \frac{\partial q_t^U}{\partial \gamma} \left(\frac{\partial z_t}{\partial q_t} + \left(\frac{\partial z_t}{\partial \alpha_t} \right) \frac{\partial \alpha_t}{\partial q_t} \right) \Big|_{q_t=q_t^U} f(z_t(q_t^U, \alpha_t | x, y)) \geq 0$$

because $\left(\frac{\partial z_t}{\partial q_t} + \left(\frac{\partial z_t}{\partial \alpha_t} \right) \frac{\partial \alpha_t}{\partial q_t} \right) \Big|_{q_t=q_t^U} \geq 0$ by (2.14) and q_t^U increases in γ . \square

This result implies that a high growth coefficient induces the firm to order more, but a higher order quantity requires more borrowing, which increases the probability of bankruptcy.

2.4.2 Liquidation Bankruptcy

Liquidation bankruptcy complicates the problem because the firm stops its operations when its ending cash position falls below zero. This problem can be formulated as a dynamic program with a stopping time. Let $\pi_t^L(q_t | x_t, y_t)$ denote the discounted expected terminal wealth period in t as a function of the state variables x_t and y_t and the action variable q_t under liquidation bankruptcy. Similarly, let $V_t^L(x_t, y_t)$ and $q_t^L(x_t, y_t)$ denote the corresponding maximum expected terminal wealth and the optimal order quantity in period t , respectively. Observe that the bankruptcy threshold demand z_t is identical to the previous section. We have the boundary condition $V_t^L(x_t, y_t) = 0$ for all $x_t < 0$ and $t \in \{1, \dots, T\}$. That is, $q_t^L(x_t, y_t) = 0$ for $x_t < 0$. In the last period, if $x_1 \geq 0$,

$$\begin{aligned} V_1^L(x_1, y_1) &\equiv \max_{q_1 \geq 0} \quad \pi_1^L(q_1 | x_1, y_1) \equiv \beta \int_{z_1}^{\infty} x_0 f(\xi_1) d\xi_1 \\ \text{s.t.} \quad x_0 &= (1 + r_f)[x_1 - c(v_1 + q_1)]^+ + py_1 + (p - s) \min\{q_1, \xi_1\} \\ &\quad + sq_1 - (1 + \alpha_1)[c(y_1 + q_1) - x_1]^+ \end{aligned}$$

$$\int_0^{z_1} \bar{F}(\xi_1) d\xi_1 \geq \frac{(c - \bar{s})q_1 - (\bar{p} - c)y_1 - x_1}{\bar{p} - \bar{s}}$$

$$\alpha_1 \geq r_f$$

$$z_1 = \left(\frac{[(1 + \alpha_1)c - s]q_1 - [p - (1 + \alpha_1)c]y_1 - (1 + \alpha_1)x_1}{p - s} \right)^+.$$

For $t > 1$ and $x_t \geq 0$,

$$V_t^L(x_t, y_t) \equiv \max_{q_t \geq 0} \quad \pi_t^L(q_t | x_t, y_t) \equiv \beta \int_{z_t}^{\infty} V_{t-1}^L(x_{t-1}, y_{t-1}) f(\xi_t) d\xi_t$$

$$\text{s.t.} \quad x_{t-1} = (1 + r_f)[x_t - c(y_t + q_t)]^+ + py_t + (p - s) \min\{q_t, \xi_t\} \\ + sq_t - (1 + \alpha_t)[c(y_t + q_t) - x_t]^+$$

$$y_{t-1} = \gamma(y_t + \min\{q_t, \xi_t\})$$

$$\int_0^{z_t} \bar{F}(\xi_t) d\xi_t \geq \frac{(c - \bar{s})q_t - (\bar{p} - c)y_t - x_t}{\bar{p} - \bar{s}}$$

$$\alpha_t \geq r_f$$

$$z_t = \left(\frac{[(1 + \alpha_t)c - s]q_t - [p - (1 + \alpha_t)c]y_t - (1 + \alpha_t)x_t}{p - s} \right)^+.$$

We first analyze a single period problem. If $x_1 \geq 0$, the firm seeks to maximize $E[\max\{x_0, 0\}]$ by choosing an order quantity. The next lemma shows that the optimal order quantity in a single period problem is equal to q_1^U if $x_1 \geq 0$.

Lemma 2.4.4 *If $x_1 \geq 0$, the optimal order quantity is $q_1^L(x_1, y_1) = q_1^U = \bar{F}^{-1}\left(\frac{c - \bar{s}}{\bar{p} - \bar{s}}\right)$. Furthermore, $V_1^L(x_1, y_1) = 0$ if $x_1 < 0$, and*

$$V_1^L(x_1, y_1) = x_1 + (\bar{p} - c)y_1 + (\bar{p} - \bar{s}) \int_0^{q_1^U} \bar{F}(\xi_1) d\xi_1 - (c - \bar{s})q_1^U$$

otherwise.

Proof: If $x_1 < 0$, then the firm is bankrupt which implies that it cannot place an order and that the value function is zero by the boundary condition. Suppose $x_1 \geq 0$, the problem formulation of a single period problem is identical

to the one we presented under reorganization bankruptcy. Hence, the classical newsvendor solution is optimal. \square

This result implies that the optimal order quantity in a single period problem is independent of the starting capital as long as the firm is not bankrupt at the beginning of the period. This separability result arises due to the risk neutral lending market, which allows us to derive the equality presented in Lemma 2.4.2. However, it no longer holds in a multi-period setting even under a risk neutral lending market. Analyzing a two-period problem illustrates the complexity that arises due to the stopping time. The objective function in a two-period problem can be written as $\pi_2^L(q_2|x_2, y_2) = \beta \int_0^\infty V_1^L(x_1, y_1) f(\xi_2) d\xi_2$. If the firm does not have bankruptcy risk, then $x_1 \geq 0$ by definition. If $q_2 > \frac{x_2 + (\bar{p} - c)y_2}{c - \bar{s}}$, then the firm has bankruptcy risk in period 2. As a result,

$$\begin{aligned} \pi_2^L(q_2|x_2, y_2, BWR) &= \beta \int_0^\infty V_1(x_1, y_1) f(\xi_2) d\xi_2 \\ &= \beta \int_{z_2}^\infty V_1(x_1, y_1) f(\xi_2) d\xi_2 \\ &= \beta \int_{z_2}^\infty V_1((p - s)(\min\{q_2, \xi_2\} - z_2), \gamma(y_2 + \min\{q_2, \xi_2\})) f(\xi_2) d\xi_2. \end{aligned}$$

The second line follows because $x_1 < 0$ for $\xi_2 < d_2$, which implies that $\int_0^{z_2} V_1(x_1, y_1) f(\xi_2) d\xi_2 = 0$. Using Lemma 2.4.4, we can write

$$\begin{aligned} \pi_2^L(q_2|x_2, y_2, BWR) &= \beta \int_{z_2}^\infty V_1((p - s)(\min\{q_2, \xi_2\} - z_2), \gamma(y_2 + \min\{q_2, \xi_2\})) f(\xi_2) d\xi_2 \\ &= \beta(p - s) \int_{z_2}^\infty (\min\{q_2, \xi_2\} - z_2) f(\xi_2) d\xi_2 \\ &\quad + \beta \int_{z_2}^\infty ((\bar{p} - c)\gamma(y_2 + \min\{q_2, \xi_2\}) + N_1(q_1^U)) f(\xi_2) d\xi_2 \\ &= \underbrace{x_2 + (\bar{p} - c)y_2 + (\bar{p} - \bar{s}) \int_0^{q_2} \bar{F}(\xi_2) d\xi_2 - (c - \bar{s})q_2}_{\text{Period 2}} \end{aligned}$$

$$+ \underbrace{\beta \int_{z_2}^{\infty} ((\bar{p} - c)\gamma(y_2 + \min\{q_2, \xi_2\}) + N_1(q_1^U))f(\xi_2)d\xi_2}_{\text{Period 1}}. (2.15)$$

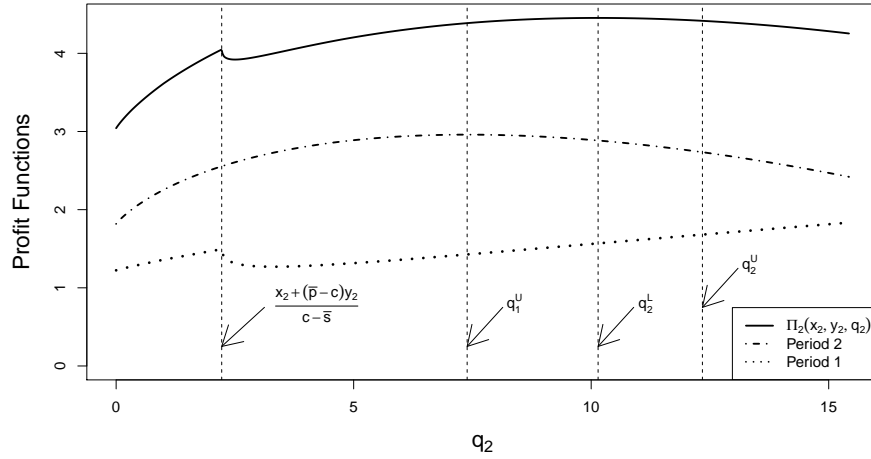
The last line follows from Lemma 2.4.2. When the firm has no bankruptcy risk, the same formula holds with $z_2 = 0$. (2.15) has two parts. The first part, $x_2 + (\bar{p} - c)y_2 + (\bar{p} - \bar{s}) \int_0^{q_2} \bar{F}_2(\xi_2)d\xi_2 - (c - \bar{s})q_2$, is a scalar (i.e., $x_2 + (\bar{p} - c)y_2$) plus the classical newsvendor function. Hence, following a myopic policy would lead to the classical newsvendor solution, q_1^U . However, q_2 also affects the second part of the profit function, $\beta \int_{z_2}^{\infty} ((\bar{p} - c)\gamma(y_2 + \min\{q_2, \xi_2\}) + N_1(q_1^U))f(\xi_2)d\xi_2$. On the one hand, q_2 has a positive impact on this part because the expression inside the integral increases in q_2 for $\gamma > 0$. That is, a high order quantity fuels growth. On the other hand, increasing q_2 increases z_2 , which indicates a higher probability of bankruptcy. Hence, the second part of the profit function is not well behaved. As a consequence, the objective function under liquidation bankruptcy is not concave or quasi-concave. Figure 2.4 illustrates these dynamics with a numerical example.

In general, q_t has a direct impact on the firm's sales revenues in period t . In addition, it affects the firm's future cash flows and bankruptcy risk because z_t , x_{t-1} , and y_{t-1} also depend on q_t . Characterizing the optimal order quantity under liquidation bankruptcy is difficult due to the non-concave nature of the objective function. However, our analysis so far indicates that the firm's order quantity should achieve the right balance between growth and bankruptcy risk. The next lemma will be useful to propose a simple heuristic that allows the firm to maintain this balance.

Lemma 2.4.5 For $t \in \{T, \dots, 2\}$,

$$a) \quad V_t^C(x, y) \leq V_t^L(x, y) \leq V_t^R(x, y) \text{ for all } (x, y).$$

Figure 2.4: The discounted expected ending cash position $\pi_2^L(q_2|x_2, y_2)$ as a function of q_2 . $x_2 = 1$ and $y_2 = 1$. $\xi_{t \in \{1,2\}}$ is log-normal with $E[\xi_t] = e^{10}$ and $\sqrt{\text{Var}(\xi_t)} = e^{10} \sqrt{e^{16} - 1}$. $\gamma = 0.25$, $p = 2$, $c = 1$, $s = 0.2$, and $r_f = 0.1$. The profit function in period 1 increases in q_2 when $q_2 \leq \frac{x_2 + (\bar{p} - c)y_2}{c - \bar{s}} = 2.22$ because the firm has no bankruptcy risk (i.e., $z_2 = 0$). When $q_2 > 2.22$, it might increase or decrease because both z_2 and $\min\{q_2, \xi_2\}$ increase in q_2 . The period 2 component of the profit function is a scalar plus the classical newsvendor function, which is maximized at $q_1^U = 7.39$. Adding these two functions leads to a non-monotone profit function. The optimal order quantity $q_2^L(x_2, y_2) = 10.12$, which is less than the unconstrained order quantity $q_2^U = 12.34$.



- b) If $x \geq 0$ and $x + (\bar{p} - c) \sum_{i=0}^{t-1} \bar{\gamma}^i y \geq (c - \bar{s}) \sum_{i=1}^t \beta^{t-i} q_i^U$ for $i \in \{t, \dots, 2\}$, then $q_i^L(x, y) = q_i^U$ for $i \in \{t, \dots, 1\}$ is an optimal policy under liquidation bankruptcy and $V_t^C(x, y) = V_t^L(x, y) = V_t^R(x, y)$.
- c) Let $\Delta_t(x, y) \equiv V_t^R(x, y) - V_t^C(x, y)$. $\Delta_t(x, y)$ decreases in x for a given y , and decreases in y for a given x .

Proof: Observe that $q_t^C(x, y)$, which avoids risky borrowing, is feasible under liquidation bankruptcy. Furthermore, if the firm follows $q_t^C(x, y)$ under liquidation bankruptcy, its expected ending cash position will be $V_t^C(x, y)$, which implies that $V_t^C(x, y) \leq V_t^L(x, y)$ because the optimal policy under liquidation bankruptcy, by definition, performs at least as good as $q_t^C(x, y)$. Similarly, ordering $q_t^L(x, y)$ is feasible under reorganization bankruptcy. Following $q_t^L(x, y)$ under reorgani-

zation bankruptcy leads to $V_t^L(x, y)$, which implies that $V_t^L(x, y) \leq V_t^R(x, y)$. This completes the first part of the proof.

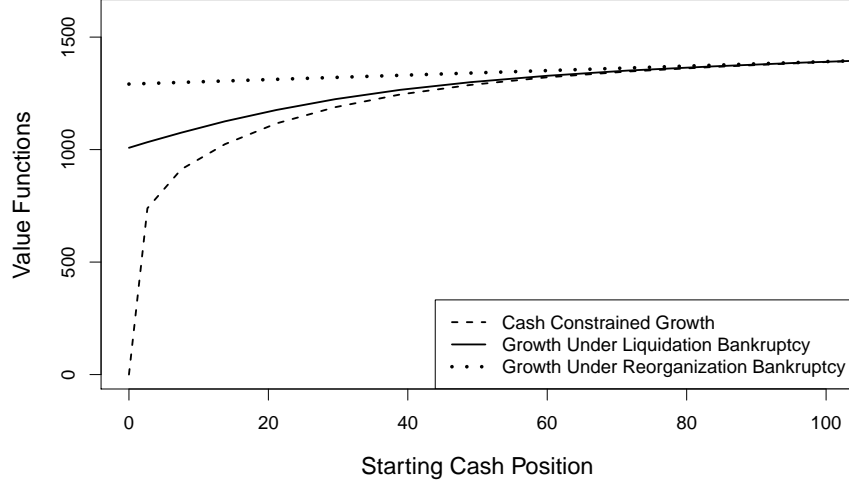
Proposition 2.3.1 shows that $q_i^C(x, y) = q_i^U$ for $i \in \{t, \dots, 1\}$ when $x \geq 0$ and $x + (\bar{p} - c) \sum_{i=0}^{t-1} \bar{\gamma}^i y \geq (c - \bar{s}) \sum_{i=1}^t \beta^{t-i} q_i^U$. Furthermore, comparing the value functions in Propositions 2.3.1 and 2.4.1 shows that $V_t^C(x, y) = V_t^R(x, y)$. Hence, part a) implies that $V_t^C(x, y) = V_t^L(x, y) = V_t^R(x, y)$, and $q_t^C(x, y)$ is an optimal policy when $x \geq 0$ and $x + (\bar{p} - c) \sum_{i=0}^{t-1} \bar{\gamma}^i y \geq (c - \bar{s}) \sum_{i=1}^t \beta^{t-i} q_i^U$.

Lemmas 2.3.1 and 2.3.2 shows that $V_t^C(x, y)$ is increasing and jointly concave in (x, y) . Furthermore, $V_t^R(x, y)$ is linear in x and y , and $V_t^R(x, y) = V_t^C(x, y)$ for large (x, y) . Hence, $\Delta_t(x, y)$ decreases in x for a given y , and decreases in y for a given x because both functions increase in x and y , but $V_t^C(x, y)$ increases at a faster rate.

□

Intuitively, the value functions of the cash constrained scenario and the reorganization bankruptcy scenario lead to a lower and an upper bound for the value function of the liquidation bankruptcy scenario, respectively. Furthermore, the gap between the value functions of the reorganization scenario and the cash constrained scenario becomes smaller as the starting cash position or the baseline demand increases. In fact, the two values functions are equal when the firm has enough cash and baseline demand to achieve unconstrained growth. This result arises because an increase in the starting cash position or the baseline demand relaxes the internal cash flow constraints. Figure 2.5 illustrates these dynamics.

Figure 2.5: The value functions $V_T^R(x_T, y_T)$, $V_T^C(x_T, y_T)$ and $V_T^L(x_T, y_T)$ as a function of starting equity x_T for $y_T = 0$. $T = 5$, $\gamma = 1.25$, $p = 5$, $c = 1$, $s = 0.2$, $r_f = 0.1$. Demand is log-normal with $E[\xi_t] = \sqrt{\text{Var}(\xi_t)} = 31.12$.



A Heuristic Solution Under Liquidation Bankruptcy. If executed properly, borrowing with risk allows the firm to achieve a higher expected ending position compared to avoiding risky borrowing. That is, following $q_t^C(x, y)$ under liquidation bankruptcy would be too conservative because it overlooks the benefit of risky borrowing. On the contrary, following q_t^R under liquidation bankruptcy would be too aggressive because q_t^R omits the firm's current state (x, y) and its bankruptcy risk. Therefore, $q_t^C(x, y)$ and q_t^R can be interpreted as two extreme policies. A linear combination of these two policies leads to a simple heuristic, $q_t^H(x, y)$. Namely, let

$$q_t^H(x, y) \equiv w_t q_t^C(x, y) + (1 - w_t) q_t^R,$$

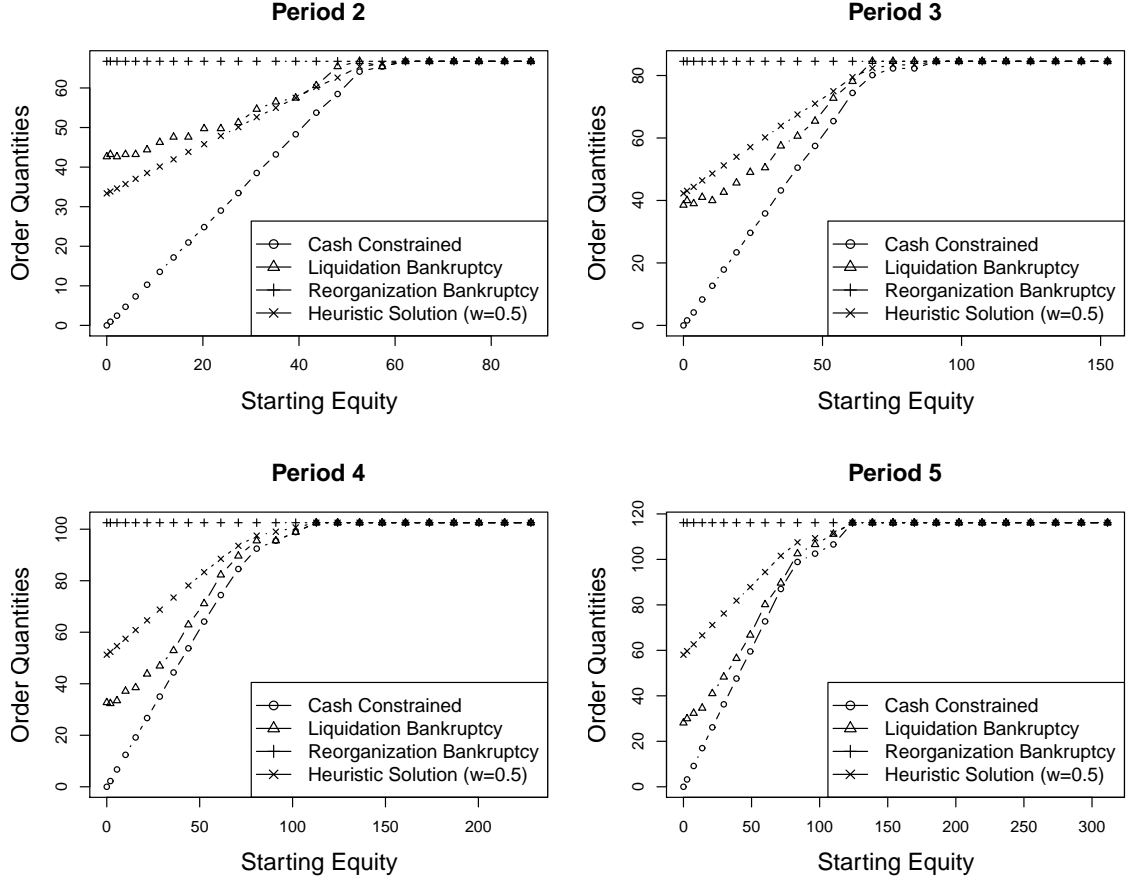
for $t \geq 2$ and $x \geq 0$, where $w_t \in [0, 1]$. Here w_t and $1 - w_t$ can be interpreted as the weights that the firm puts on conservative and aggressive growth, respectively. We propose five alternatives to choose w_t .

- Heuristic 1: Set $w_t = 1$, which mimics a conservative growth policy.

- Heuristic 2: Set $w_t = 0$, which mimics an aggressive growth policy.
- Heuristic 3: Set $w_t = 0.5$, which takes a simple average of $q_t^C(x, y)$ and q_t^R to balance the tradeoff between growth and risk.
- Heuristic 4: Set $w_t = \mathbb{P}_t(q_t^U|x, y)$. This heuristic policy takes the bankruptcy probability under reorganization bankruptcy into account by setting the weight of q_t^R equal to the probability of surviving at the end of period t after ordering q_t^R under reorganization bankruptcy. A high $\mathbb{P}_t(q_t^U|x, y)$ leads to a higher weight on $q_t^C(x, y)$, which lowers the firm's risk exposure.
- Heuristic 5: Set $w_t = \frac{V_t^C(x, y)}{V_t^C(x, y) + V_t^R(x, y)}$. This heuristic sets the weights according to the value functions of the conservative growth and the reorganization bankruptcy scenarios. Lemma 2.4.5 implies that the firm puts more weight on q_t^R when (x, y) is small.

In addition to the optimal order quantities of the conservative growth, growth under liquidation bankruptcy, and growth under reorganization bankruptcy scenarios, Figure 2.6 shows the heuristic solution $0.5q_t^C(x, y) + (1 - 0.5)q_t^R$ as a function of the firm's starting cash position. Intuitively, one would expect Heuristics 3, 4 and 5 to achieve higher expected ending cash positions under liquidation bankruptcy than Heuristics 1 and 2 (i.e., $q_t^C(x, y)$ and q_t^R) because 3, 4 and 5 should provide better approximations for $q_t^L(x, y)$ by balancing the tradeoff between growth and risk. In the next section, we perform numerical experiments to demonstrate the validity of this intuition.

Figure 2.6: The optimal order quantities $q_t^C(x_t, y_t)$, $q_t^L(x_t, y_t)$, q_t^R , and the heuristic solution for the liquidation bankruptcy case, $q_t^H(x_t, y_t) = 0.5q_t^C(x_t, y_t) + 0.5q_t^R$, as a function of starting equity x_t for $y_t = 0$ in periods $t \in \{2, \dots, 5\}$. $T = 5$, $\gamma = 1.25$, $p = 5$, $c = 1$, $s = 0.2$, $r_f = 0.1$. Demand is log-normal with $E[\xi_t] = \sqrt{\text{Var}(\xi_t)} = 31.12$.



2.5 Numerical Analysis

Our main objectives in this section are to measure the impact of model parameters on the value functions under the conservative growth, growth with liquidation bankruptcy and growth with reorganization bankruptcy scenarios and to evaluate the performance of the five heuristics proposed in the previous section in solving the firm's problem under liquidation bankruptcy. We also take a deeper look at the evolution of the firm's bankruptcy risk over time, and show

that the firm is more likely to fail in the early periods, regardless of the policy used. Lastly, we present a simple decision theoretic framework to capture the tradeoff between growth and bankruptcy risk. Such a framework can be used to introduce risk aversion to our model.

We define nine different cases with varying model parameters. Table 2.1 shows the parameter values. In each case, we vary one parameter to measure its impact on the value functions. For example, Cases 1, 2, and 3 capture the impact of the time horizon. Similarly, Cases 1, 4 and 5 capture the impact of the firm's growth prospects, as measured by γ , whereas Cases 1, 6 and 7 focus on the impact of demand uncertainty. The coefficient of variation of ξ (i.e., $\sqrt{\text{Var}(\xi)}/E[\xi]$) equals 0.5 in Case 6, and equals 2 in Case 7. It is equal to 1 in the remaining cases. The expected demand, $E[\xi_t]$, equals 100 in all cases. Lastly, Cases 1, 8 and 9 capture the impact of the profit margin. We fix p , c , and vary s to vary the profit margin, which we define as $\frac{\bar{p}-c}{\bar{p}-s}$.

Table 2.1: Nine different cases and the corresponding value functions. Common parameters are $p = 1$, $c = 0.8$, $r_f = 0.05$, and $(x_T, y_T) = (1, 0)$. Demand is i.i.d. log-normal for $t \in \{T, \dots, 1\}$.

Case	Parameters					Value Functions			
	T	γ	$E[\xi_t]$	$\sqrt{\text{Var}(\xi_t)}$	s	V_T^C	V_T^L	V_T^R	V_T^R/V_T^C
1	5	0.1	100	100	0.65	15.68	27.69	30.49	1.94
2	3	0.1	100	100	0.65	8.88	17.44	19.18	2.16
3	10	0.1	100	100	0.65	38.13	50.11	54.36	1.43
4	5	0.6	100	100	0.65	27.17	50.96	59.47	2.19
5	5	1.1	100	100	0.65	50.14	105.84	136.56	2.72
6	5	0.1	100	50	0.65	20.27	46.87	48.32	2.38
7	5	0.1	100	200	0.65	10.42	13.17	15.54	1.49
8	5	0.1	100	100	0.60	12.36	25.19	27.98	2.26
9	5	0.1	100	100	0.70	20.97	31.24	34.00	1.62

We first make some simple observations from Table 2.1. The firm's expected ending cash position increases in the length of the time horizon, the

firm's growth prospects and the profit margin, and decreases in demand uncertainty, regardless of the scenario (i.e., conservative, liquidation or reorganization). These observations are intuitive because the firm is likely to sell more units as T or γ increases and makes more money per unit when the profit margin is high. Lastly, an increase in the demand coefficient of variation increases the firm's risk exposure, which decreases its expected ending cash position.

A comparison of the value functions enables us to assess the monetary benefit of being able to borrow with risk under reorganization bankruptcy. One way to measure such a benefit is to compute the ratio of two value functions. For example, we find that V_T^R/V_T^C increases in the firm's growth prospects, as measured by γ . This result shows that the benefit of leverage is more pronounced for growth firms. It arises because q_t^R increases in γ . On the other hand, $q_t^C(x_t, y_t)$ remains constant if the credit limit is binding. As a result, the difference between q_t^R and $q_t^C(x_t, y_t)$ increases, which leads to an increase in V_T^R/V_T^C . Similar inferences can be drawn regarding the relative advantages and drawbacks of liquidation bankruptcy by computing V_T^L/V_T^C and V_T^L/V_T^R , respectively.

We rely on various heuristics to solve the firm's growth problem under liquidation bankruptcy. We first compute the value functions, $V_t^C(x, y)$, $V_t^L(x, y)$ and $V_t^R(x, y)$, and the corresponding optimal order quantities, $q_t^C(x, y)$, $q_t^L(x, y)$ and q_t^R , for $t \in \{T, \dots, 1\}$ for a given set of parameter values. This step allows us to obtain the five heuristic solutions. Then we generate 2500 sample paths for each case presented in Table 2.1, and compute the firm's ending cash position on each sample path using the optimal policy as well as the five heuristics under liquidation bankruptcy. Table 2.2 illustrates the performance of the optimal solution in our simulation study. Tables 2.3,...,2.7 show the performance of Heuristics

1,...,5, respectively.

Table 2.2: Simulation analysis of the optimal discounted expected ending cash position and the corresponding probability of bankruptcy under the nine cases presented in Table 2.1. We generate 2500 sample paths, and track the firm's cash flow and baseline demand under the optimal policy. Common parameters are $p = 1$, $c = 0.8$, $r_f = 0.05$, and $(x_T, y_T) = (1, 0)$. Please see Table 2.1 for the model parameters of each case.

Case	Optimal Solution, q_t^L	
	$\beta^T E[x_0^+]$	$Pr(x_0 < 0)$
1	27.55	0.09
2	17.57	0.12
3	50.25	0.05
4	50.64	0.09
5	104.78	0.12
6	47.05	0.02
7	13.12	0.15
8	25.09	0.10
9	31.12	0.06

Table 2.3: Simulation analysis of Heuristic 1 under the nine cases presented in Table 2.1. Please see Table 2.1 for the model parameters of each case. Table 2.2 presents the details of our simulation study.

Case	Heuristic 1, q_t^C		
	$\beta^T E[x_0^+]$	$Pr(x_0 < 0)$	Performance
1	16.89	0.00	61.31%
2	9.08	0.00	51.69%
3	40.45	0.00	80.50%
4	29.01	0.00	57.28%
5	51.33	0.00	48.98%
6	21.04	0.00	44.73%
7	10.57	0.00	80.58%
8	13.12	0.00	52.28%
9	21.86	0.00	70.24%

Our analysis indicates that Heuristics 1 and 2 (i.e., ordering $q_t^C(x, y)$ and q_t^R , respectively) fail to maintain the balance between growth and bankruptcy risk. To see this, observe that the probability of bankruptcy under the optimal policy is always greater than the one under Heuristic 1, which is always equal to zero,

Table 2.4: Simulation analysis of Heuristic 2 under the nine cases presented in Table 2.1. Please see Table 2.1 for the model parameters of each case. Table 2.2 presents the details of our simulation study.

Case	Heuristic 2, q_t^R		
	$\beta^T E[x_0^+]$	$Pr(x_0 < 0)$	Performance
1	25.05	0.25	90.94%
2	16.78	0.24	95.51%
3	42.28	0.26	84.14%
4	39.64	0.41	78.28%
5	30.80	0.83	29.40%
6	45.85	0.07	97.45%
7	11.96	0.36	91.20%
8	22.95	0.26	91.50%
9	27.71	0.25	89.03%

Table 2.5: Simulation analysis of Heuristic 3 under the nine cases presented in Table 2.1. Please see Table 2.1 for the model parameters of each case. Table 2.2 presents the details of our simulation study.

Case	Heuristic 3, $(q_t^C + q_t^R)/2$		
	$\beta^T E[x_0^+]$	$Pr(x_0 < 0)$	Performance
1	27.53	0.06	99.92%
2	17.12	0.05	97.41%
3	50.08	0.06	99.65%
4	50.55	0.12	99.81%
5	97.67	0.30	93.21%
6	44.47	0.00	94.52%
7	13.05	0.19	99.49%
8	24.98	0.07	99.58%
9	31.02	0.05	99.69%

and less than the one under Heuristic 2. That is, $q_t^C(x, y)$ is too conservative, and misses profitable growth prospects, whereas q_t^R is too aggressive, and overlooks bankruptcy risk. As a result, both heuristics perform relatively poorly. Heuristic 1's performance varies between 44.73% and 80.50% of the optimal solution in the nine cases we analyze. Similarly, Heuristic 2's performance varies between 29.40% and 97.45% of the optimal solution. On the contrary, Heuristics 3, 4 and 5 perform relatively well. For instance, they all achieve more than 95% of the

Table 2.6: Simulation analysis of Heuristic 4 under the nine cases presented in Table 2.1. Please see Table 2.1 for the model parameters of each case. Table 2.2 presents the details of our simulation study.

	Heuristic 4, $q_t^C + Pr(x_{t-1} \geq 0)(q_t^R - q_t^C)$		
	$\beta^T E[x_0^+]$	$Pr(x_0 < 0)$	Performance
1	26.72	0.17	96.99%
2	17.46	0.15	99.35%
3	46.64	0.16	92.81%
4	49.54	0.17	97.82%
5	91.45	0.19	87.28%
6	46.37	0.05	98.56%
7	12.64	0.29	96.38%
8	24.46	0.18	97.51%
9	29.70	0.17	95.43%

Table 2.7: Simulation analysis of Heuristic 5 under the nine cases presented in Table 2.1. Please see Table 2.1 for the model parameters of each case. Table 2.2 presents the details of our simulation study.

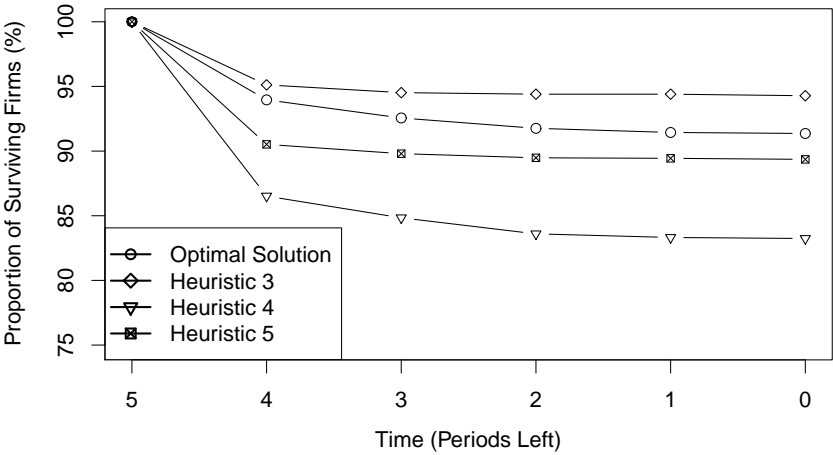
Case	Heuristic 5, $q_t^C + \frac{V_t^R}{V_t^C + V_t^R}(q_t^R - q_t^C)$		
	$\beta^T E[x_0^+]$	$Pr(x_0 < 0)$	Performance
1	27.38	0.11	99.40%
2	17.52	0.11	99.69%
3	49.70	0.08	98.90%
4	48.04	0.21	94.86%
5	73.79	0.55	70.43%
6	46.45	0.01	98.74%
7	12.98	0.22	98.97%
8	24.84	0.12	99.04%
9	30.76	0.09	98.84%

optimal expected ending cash position in seven out of nine cases. A relatively high performance of these three heuristics arises due to their ability to balance the tradeoff between growth and bankruptcy risk. Surprisingly, Heuristic 3, which takes a simple average of $q_t^C(x, y)$ and q_t^R , performs better than Heuristics 4 and 5 in seven cases. Furthermore, it achieves 97.41% and 94.52% of the optimal solution in the other two cases.

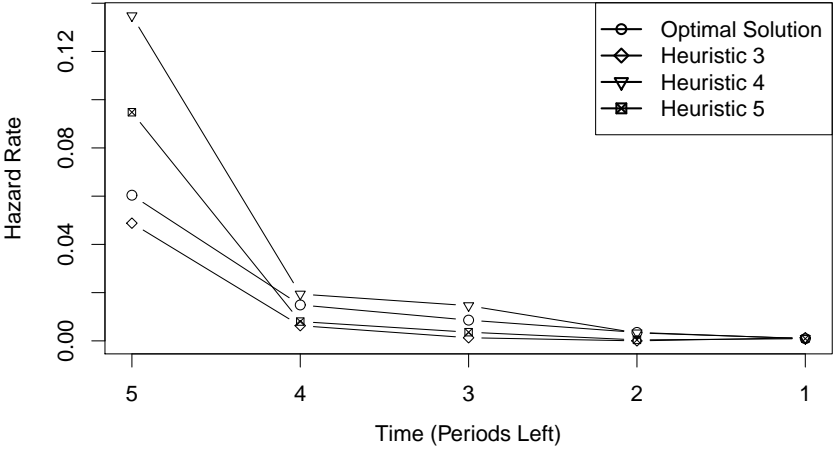
Another important aspect of bankruptcy risk is its evolution over time. Figure 2.7(a) shows the proportion of sample paths in which the firm survives, and Figure 2.7(b) shows the hazard rate (i.e., the number of sample paths in which the firm declares bankruptcy in period t divided by the number of sample paths in which the firm survives at the beginning of period t) in Case 1 under different policies. Both figures illustrate that the firm is more likely to fail in the early periods, regardless of the policy used. This result arises due to a few reasons. First, the firm needs to make larger new investments (i.e., choose high q_t values) in the early periods to fuel growth. For example, (2.5) shows that q_t^R decreases over time. Second, demand uncertainty measured by the coefficient of variation of D_t , i.e., $\sqrt{\text{Var}(\xi_t)}/(y_t + E[\xi_t])$, is high in the early periods because the firm has no or very little baseline demand. Combining large investments and high demand uncertainty with cash flow constraints puts the firm into financial distress. If the firm survives through the first few periods, its new investments and demand uncertainty go down. Furthermore, it accumulates cash and/or baseline demand. As a result, bankruptcy becomes less likely. Thus, we illustrate that bankruptcy risk in our model is driven by new investments, demand uncertainty, and the firm's cash position and baseline demand.

Figure 2.8 shows the relationship between the probability of bankruptcy and the discounted average ending cash position over all sample paths in which the firm survives, i.e., $\beta^T E[x_0 | x_0 \geq 0]$. The main takeaway of Figure 2.8 is that for a given case, a policy that leads to a high probability of bankruptcy also leads to a high expected ending cash position given that the firm survives. This observation is consistent with the anecdotal evidence we present in the introduction. That is, on average, firms following aggressive growth policies are more likely to fail (e.g., The J. Peterman Co.) compared to firms following conser-

Figure 2.7: The proportion of sample paths in which the firm survives and the corresponding hazard rate. We analyze Case 1 under the optimal policy and three heuristic solutions. See Tables 2.1 for the model parameters.



(a)



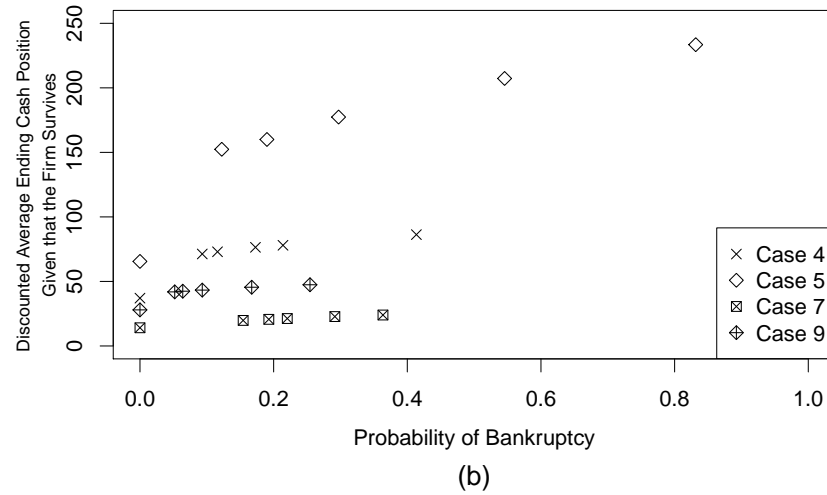
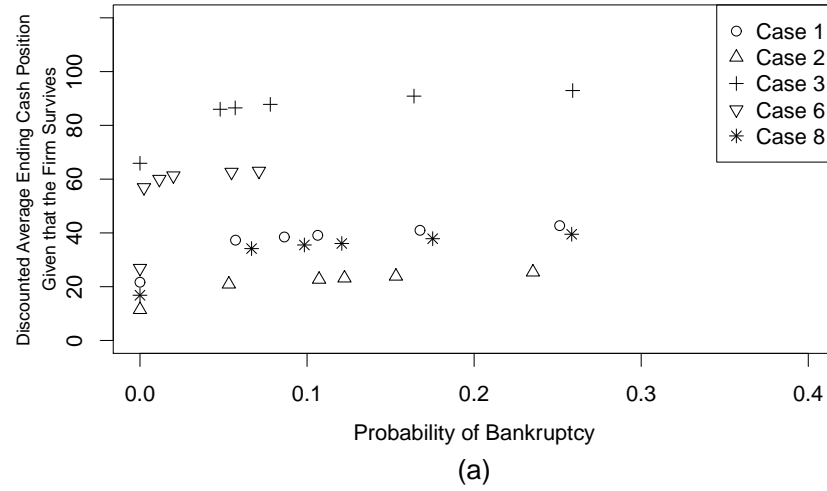
(b)

vative growth policies (e.g., Ingres), but those that follow aggressive growth policies and survive achieve tremendous success (e.g., Oracle). In this regard, the final outcome of each policy can be mapped into a simple decision tree with two nodes. The first and the second nodes pay 0 and $\beta^T E[x_0 | x_0 \geq 0]$ with probabilities $Pr(x_0 < 0)$ and $Pr(x_0 \geq 0)$, respectively. In this decision theoretic view of the firm's problem, a high expected payoff upon survival (i.e., a high $\beta^T E[x_0 | x_0 \geq 0]$ value) is associated with a low survival probability, $Pr(x_0 \geq 0)$. This approach can be used to formulate the firm's problem under risk aversion because the firm can be modeled as an expected utility maximizer seeking to maximize $\beta^T E[x_0^+]$ (or $\beta^T E[x_0 | x_0 \geq 0]$) minus the probability of bankruptcy times a bankruptcy cost. Similar utility functions are used in the literature to assess firm performance under bankruptcy risk (e.g., [78] and references therein).

2.6 Limitations and Future Work

In this chapter, we provide an operations management perspective on the relationship between growth and bankruptcy risk for a cash constrained firm. Our analysis shows that in the absence of bankruptcy risk and cash flow constraints, the firm overinvests (i.e., orders more than the classical single-period newsvendor solution) to fuel growth. In addition, comparing the unconstrained and the constrained growth scenarios without bankruptcy risk indicates that internal cash flow constraints can significantly depress firm growth. We also show that the presence of cash flow constraints forces the firm to make financially risky investment decisions. We present a closed form solution for the firm's problem under reorganization bankruptcy, and propose simple heuristics that balance the tradeoff between growth and bankruptcy risk under liquidation bankruptcy.

Figure 2.8: The probability of bankruptcy and the discounted average ending cash position over all sample paths in which the firm survives (i.e., $\beta^T E[x_0 | x_0 \geq 0]$). We analyze nine different scenarios under six different ordering policies (i.e., the optimal solution and the five heuristic solutions). Each data point represents a case-policy pair. See Tables 2.1 and 2.2 for the model parameters of each scenario and the formulation of each heuristic, respectively.



Our model has some limitations. First, unlike the mainstream inventory theory models, the firm in our model does not carry inventory from one period to the next. Although carrying inventory does not lead to any major changes in the firm's operational decisions, it complicates the bankruptcy process and the firm's interactions with financial markets. Future research may examine the impact of inventories and other assets on growth and bankruptcy risk. Second, we focus on maximizing the firm's expected ending cash position. Other objectives, such as maximizing the net present value of expected dividends, can also be considered. Future theoretical research may also examine alternative lending models and competition among multiple firms. Lastly, future empirical research may focus on identifying the operational determinants of firm growth.

CHAPTER 3

EXPLAINING THE PROBABILITY OF BANKRUPTCY USING INVENTORY TURNOVER

3.1 Introduction

We analyze the relationship between inventory productivity and financial distress for retailers. There are a few reasons that might lead to a negative relationship between inventory productivity and financial distress. First, retailers' working capital needs are driven by their inventory investments because inventory is usually the largest current asset in a retailer's books. That is, high inventory productivity means relatively less inventory, which means less working capital. Second, high inventory productivity leads to a relatively shorter cash cycle, which implies a faster turnaround from procurement to cash. As a result, retailers with high inventory productivity are less likely to face cash flow problems. Lastly, high inventory productivity means lower excess inventory and stock-out risks because retailers with high inventory productivity have more flexibility to absorb demand shocks.

Quantifying the relationship between inventory productivity and bankruptcy risk is important from a financial standpoint because incorporating inventory information into the existing bankruptcy prediction models can improve model accuracy. As a result, better estimates can be generated regarding the future of a distressed firm. Quantifying this relationship is also important from an operations management standpoint because the existing inventory models can be modified to take bankruptcy risk and its financial implications into account.

Financial distress often leads to corporate failures, which have costly con-

sequences for all stakeholders including owners or shareholders, employees, creditors, customers, and suppliers. According to a general definition stated in Dimitras et al. [27], a failure is a situation in which a firm cannot pay lenders, preferred stock shareholders, suppliers, etc. or it is bankrupt according to the law. Modeling firm failures, especially bankruptcies, has been widely studied in accounting, economics and finance since the early 1960s. The majority of the empirical work in these areas use financial ratios to quantify financial distress. The ratios that explain financial distress include net income to total assets, total liabilities to total assets, and working capital to total assets.

While the ratio based bankruptcy prediction models are useful in terms of providing early warning signals for a potential bankruptcy, they provide very little insights regarding how to save a distressed firm. That is, they are descriptive, not prescriptive. For example, past research shows that a high ratio of total liabilities to total assets is a strong bankruptcy predictor. However, recommending a distressed firm to lower its debt to avoid bankruptcy would be an empty statement because a high leverage is not the main problem; it is a symptom of more fundamental problems. Hence, the existing bankruptcy prediction models are more useful to creditors seeking to assess their risk than to managers. Consequently, identifying the root causes of financial distress and taking actions to eliminate them would be a more fruitful direction from a turnaround management standpoint.

One could hypothesize many potential root causes for financial distress such as cash flow problems, organizational issues, incapable management, competitive pressure, and unsuccessful pricing and marketing strategies. Such a list also would also include poor operational decisions. In fact, a growing body of

literature in operations management has focused on the relationship between a firm's operational decisions and its bankruptcy risk. Xu and Birge [82], Buza-cott and Zhang [15], Dada and Hu [26], and Alan and Gaur [2] study a cash constrained firm's interactions with a lender, and characterize the firm's operational and financial decisions at equilibrium. Financial considerations and bankruptcy risk also affect a firm's survival strategy (Archibald et al. [6]), capacity investment decisions (Swinney et al. [78]), choice of production technologies (Lederer and Singhal [53] and Boyabatli and Toktay [13]), and decisions to offer an IPO (Babich and Sobel [9]), issue dividends (Hu et al. [43] and references therein), and shut down (Xu and Birge [83]). Despite these theoretical studies, to the best of our knowledge, no empirical evidence exists regarding the impact of a firm's operational decisions on its bankruptcy risk.

In this chapter, we focus on retail bankruptcies, and test whether inventory productivity is related to the probability of bankruptcy. Many measures of inventory productivity are used in practice, including inventory turnover, gross margin return on inventory, and the sales to inventory ratio. We use inventory turnover as our inventory productivity metric. Our data set contains large retailer bankruptcy filings (i.e., retailers with \$100M or more in assets at the time of bankruptcy) between 1980-2010. Our analysis shows that firms with high inventory turnover have lower probability of bankruptcy and that adding inventory turnover as an explanatory variable improves the model fit of three commonly used bankruptcy models. These results are important to illustrate the relevance of operational efficiency in explaining and managing financial distress.

The rest of the chapter is organized as follows. Section 3.2 provides a lit-

erature review on the existing bankruptcy prediction models. We develop our hypothesis in Section 3.3. Sections 3.4 and 3.5 provide details about our data set and empirical analysis, respectively. Section 3.6 summarizes the limitations of our model and data set, and outlines future work.

3.2 Literature Review

In this section, we present the commonly used techniques to model and predict bankruptcies. Altman [4], in a seminal paper on bankruptcy modeling, proposes discriminant analysis (DA) to identify the most important variables in detecting bankruptcy risk. In general, DA is used to determine which covariates discriminate between two or more groups. Since we have two distinct groups (i.e. bankrupt and non-bankrupt), the analysis can be transformed into a simple discriminant function of the form $z_i = \beta X_i$, where z_i is the z -score for firm i and X_i is a vector of firm specific covariates. The goal is to estimate β such that every firm is correctly identified as bankrupt or non-bankrupt based on its z -score. A cutoff score is computed based on the initial information and the cost of misclassification (i.e., identifying a bankrupt firm as non-bankrupt and vice-versa), and a firm is classified as bankrupt or non-bankrupt based on its z -score and the cut-off score. Altman [4] uses a sample of 66 manufacturing firms, of which 33 declared bankruptcy during the period 1964-1965, and the remaining 33 firms were in operation in 1966. The latter 33 firms were chosen on a stratified random basis to construct a matched pair sample. The final DA, which is known as the z -score model, is

$$z = 1.2x_1^z + 1.4x_2^z + 3.3x_3^z + 0.6x_4^z + x_5^z,$$

where $x_1^z := \text{Working Capital} / \text{Total Assets}$, $x_2^z := \text{Retained Earnings} / \text{Total Assets}$, $x_3^z := \text{Earnings Before Interest and Taxes} / \text{Total Assets}$, $x_4^z := \text{Market Value of Equity} / \text{Book Value of Debt}$, and $x_5^z := \text{Sales} / \text{Total Assets}$. The cut-off score minimizing the total number of mis-classifications was 2.675. That is, the original z -score model would classify a firm as bankrupt (healthy) if its z -score was below (above) 2.675. In the following years, Altman has introduced alternative (DA) models (e.g., the ZETA score model), which generate better model fit and more accurate out of sample predictions. Please see Altman and Hotchkiss [5] and references therein for details regarding DA models.

Eisenbass [33] summarizes the pitfalls of DA. DA assumes that the covariates are independent and follow a multivariate normal distribution within each group. However, financial ratios are highly correlated, and in most cases, covariates are right skewed, which make the validity of these assumptions highly questionable. Dimitras et al. [27] argue that logit/probit models are preferable to DA because they are more flexible in terms of handling various types of covariates including binary and categorical variables. In the logit model, the bankruptcy probability for firm i can be written as

$$Pr(X_i; \beta) = \frac{1}{1 + \exp(-\beta X_i)}.$$

In terms of classifying firms as bankrupt or non-bankrupt, logarithm of the odds of the above formula, $\log\left(\frac{Pr(X_i; \beta)}{1 - Pr(X_i; \beta)}\right)$, corresponds to the z -score of a DA model. Similar to DA, a cut-off probability that minimizes the type I and type II errors is determined, and a firm is classified as bankrupt (non-bankrupt) if its probability of bankruptcy is greater (less) than the cutoff probability. Ohlson [67] uses a logit model for bankruptcy prediction. He concludes that firm size, financial structure and performance, and current liquidity are the most important factors that determine the probability of bankruptcy. Logit/probit models are also used

by practitioners. For example, Moody's RiskCalc uses a probit model to generate expected credit default frequencies for private firms. Although this model is used to analyze credit worthiness and quality, its results also serve as a proxy for bankruptcy risk. See Falkenstein et al. [34] for details.

Survival Analysis (SA) is another commonly used technique for bankruptcy modeling. Kiefer [46] provides the theoretical basis for applying survival analysis on econometric problems. Shumway [75] shows that single period models give biased and inconsistent bankruptcy estimates, and uses SA for bankruptcy modeling. He illustrates the link between survival models and multi-period logit models. His model includes a combination of accounting ratios and market-driven variables, and produces more accurate out of sample forecasts compared to the aforementioned models. He shows that some market driven variables such as market size, past stock returns and the idiosyncratic standard deviation of stock returns are statistically significant bankruptcy predictors. Chava and Jarrow [21] extend Shumway [75] by incorporating industry effects. We provide more details regarding these models in Section 3.5.

Tools originating from mathematical finance are also used in bankruptcy prediction. For example, Merton [60] introduces an option theoretic framework. He models the firm's total asset value as a European call option because equity holders have limited liability for debt payments in the event of a bankruptcy. He defines the face value of the firm's liabilities as the strike price of the call option, which expires when the debt matures. At the debt maturity date, if assets are higher than liabilities, then the equity holders exercise their option and their pay off is what is left after paying debt holders. Otherwise, they let the option expire. As a result, the firm files for bankruptcy and the payoff to the equity

holders is zero. Other papers following similar modeling approaches include Campbell et al. [17], Duffie et al. [30], Duffie [29] and references therein.

The aforementioned models have a common drawback from an operations management standpoint. They focus on financial ratios and market variables, which cannot reveal the impact of a firm's operational characteristics and decisions on its bankruptcy risk. Once the relationship between operational decisions and bankruptcy risk is quantified, it might be possible to construct more accurate bankruptcy prediction models.

3.3 Motivation

Ideally, we would like to establish a direct link between a retailer's operational characteristics and decisions (e.g., inventory management capabilities and order quantities, respectively) and its bankruptcy risk. While establishing such a relationship is possible in theory, it is not a straightforward task in an empirical study because most operational decisions are not reported in financial statements. This forces us to find a proxy for a retailer's operational performance. Inventory turnover and other related metrics (e.g., sales to inventory ratio) are commonly used in the literature to study the relationship between a firm's operational characteristics and financial performance (Alan et al. [3] and references therein).

We use inventory turnover as a proxy for a firm's operational efficiency, and hypothesize a negative relationship between inventory turnover and bankruptcy risk. Intuitively, this relationship arises because holding less inventory decreases the possibility of having unsold units, which increases the

retailer's inventory turnover. Furthermore, relatively less inventory means less working capital, faster turnaround from procurement to cash, and more flexibility to absorb demand shocks. As a result, a high inventory turnover retailer's financial distress should be minimal. These dynamics lead to our hypothesis:

Hypothesis 3.3.1 *Retailers with high inventory turnover have lower probability of bankruptcy.*

We test this hypothesis after describing our data set.

3.4 Data

We use the UCLA-LoPucki Bankruptcy Research Database to test the hypothesis of a negative relationship between inventory turnover and financial distress. This database includes all those bankruptcy cases filed under Chapter 7 or Chapter 11, in which the firm had assets of \$100 million as of the last 10-K filed prior to bankruptcy. In this database, assets are measured in 1980 dollars. That is, a bankrupt firm's assets as of the last 10-K filed prior to bankruptcy are discounted to 1980 dollars using the Consumer Price Index for All Urban Consumers.¹ LoPucki [57] presents the rules defining a bankruptcy case. We use only a subset of this data set, which includes retail bankruptcies. Furthermore, we exclude two retail segments, *Eating and Drinking Places* and *Automotive Dealers and Service Stations*, because these businesses have more significant service components than inventory management components. Our data set includes 58 retail bankruptcies filed between 1980 and 2010. Table 3.1 gives a classification of bankruptcies by their retail segments. Retailers are grouped using the two digit SIC codes.

¹The Consumer Price Index for All Urban Consumers is available at <http://data.bls.gov/cgi-bin/surveymost?bls>

Table 3.1: Classification of bankruptcy filings based on the two digit SIC codes.

Two Digit SIC Code	Segment Description	Number (%) of Bankruptcies
53	General Merchandize Stores	18 (31.03%)
54	Food Stores	10 (17.24%)
56	Apparel and Accessory Stores	9 (15.52%)
57	Furniture and Home Furnishing Stores	8 (13.79%)
59	Miscellaneous Retail	13 (22.41%)
Total		58 (100%)

Figure 3.1 shows the distribution of firm ages at the time of bankruptcy. Following Chava and Jarrow [21], we define firm age as the difference between the year in which firm declared bankruptcy and the year in which the firm went public. Figure 3.1 illustrates that most bankruptcies occur in the early years of a firm's existence. The average and the median age at the time of bankruptcy are 10.50 and 14.89, respectively. Table 3.2 presents some recent high profile bankruptcy cases.

Figure 3.1: Firm age at the time of bankruptcy. Firm age corresponds to the difference between the year in which the firm declares bankruptcy and the first year in which the firm is publicly traded.

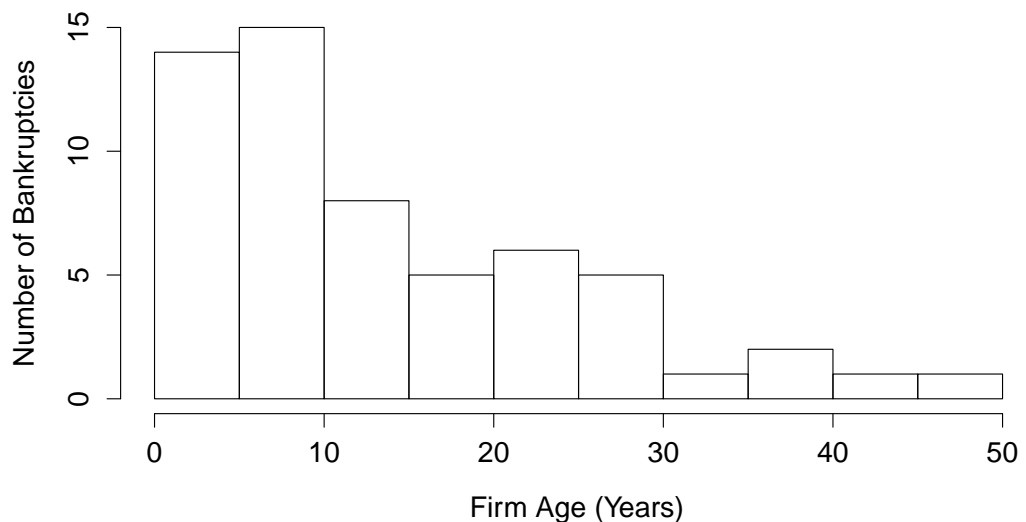


Table 3.2: Some high profile bankruptcy filings between 2000 and 2010.

Name	SIC Code	Filing Date
Kmart Corp.	53	1/22/2002
The Great Atlantic & Pacific Tea Company, Inc.	54	12/12/2010
Eddie Bauer Holdings, Inc.	56	6/17/2009
Circuit City Stores, Inc.	57	11/10/2008
WebVan Group, Inc.	59	7/13/2001

We merge the LoPucki database with the Compustat annual database using CUSIP, which is a unique identification number. The corresponding Compustat data set includes retailers from two digit SIC codes 53, 54, 56, 57, and 59 that have \$100 million or more measured in 1980 dollars, as of the last 10-K filed. Our augmented data set has $n = 339$ retailers. The total number of firm year observations is 4092.

We lag Compustat data so that they are observable by the market at the beginning of each calendar year. We substitute missing data with the previous available observation. To be consistent with the literature (e.g., Shumway [75]), we winsorize² the accounting data at the 1 and 99 percentiles. Table 3.3 shows the average and the median inventory values and the average inventory to current assets and inventory to total assets ratios in our augmented data set. It indicates that more than half of a large retailer's current assets are inventories, on average.

²Winsorization is a commonly used data transformation technique, which reduces the impact of outliers by setting all outliers equal to a specific percentile of the data. For example, a 98% winsorization sets all data points below the 1st percentile equal to the 1st percentile and all data points above the 99th percentile equal to the 99th percentile.

Table 3.3: The average and the median inventory values as well as the average inventory to current assets (ICA) and the average inventory to total assets (ITA) ratios. We compute the average and median inventory values over all firm-year observations (i.e., 4092 data points) after discounting inventories to 1980 dollars using the Consumer Price Index for All Urban Consumers.

SIC Code	Average Inventory (\$M)	Median Inventory (\$M)	Average ICA Ratio	Average ITA Ratio
53	1074	468	63.66%	34.98%
54	514	179	59.77%	22.62%
56	315	158	56.26%	32.45%
57	408	169	59.18%	38.26%
59	447	165	56.89%	35.43%
All Segments	557	193	58.96%	32.42%

3.5 Analysis and Results

3.5.1 Empirical Model

Our model specification is identical to that of Chava and Jarrow [21]. We observe a total of $n = 339$ firms at discrete time points $t = 1980, 1981, \dots, 2010$. Let $T = 2010$, and let τ_i denote the time of bankruptcy for firm i . Let N_{it} be an indicator variable, which is equal to 1 if $\tau_i \leq t$ and 0 otherwise. Furthermore, let t_i denote the first time at which firm i is observed. By definition, $t_i = 1980$ for firms that are actively traded in 1980, and $t_i > 1980$ for firms that enter the data set after 1980. Similarly, let T_i denote the last period in which firm i is observed. $T_i = \tau_i \leq T$ if the firm declares bankruptcy in τ_i . T_i could also be equal to T if the firm survives at the end of our study. If a firm remains in the data set after declaring bankruptcy, we only include the firm year observations up to (including) the first bankruptcy date in our data set and omit N_{it} and X_{it} for $t > \tau_i$. A firm that leaves the data set without declaring bankruptcy has $N_{it} = 0$ for $t \in \{t_i, \dots, T_i\}$. $T_i < T$ with $\tau_i = \infty$ is also possible if the firm leaves the data set

without declaring bankruptcy. Such a censoring could occur due to a merger or an acquisition.

Let X_{it} denote the time varying covariates of firm i at time t . Discrete time conditional hazard rate can be defined as

$$P_{it} = \Pr(\tau_i = t | \tau_i \geq t, X_{i,t_i}, X_{i,t_i+1}, \dots, X_{i,T_i}) \quad \text{for } t_i + 1 \leq t \leq T_i. \quad (3.1)$$

Chava and Jarrow [21] show that the log-likelihood function corresponding to (3.1) is equal to

$$\begin{aligned} & \log L(N_{1,T_1}, \dots, N_{n,T_n} | X_{1,t_1}, \dots, X_{1,T_1}; \dots; X_{n,t_n}, \dots, X_{n,T_n}) \\ &= \sum_{i=1}^n \sum_{t=t_i+1}^{T_i} [N_{it} - N_{i,t-1}] \log\left(\frac{P_{it}}{1 - P_{it}}\right) + \sum_{i=1}^n \sum_{t=t_i+1}^{T_i} \log(1 - P_{it}) \\ &= \sum_{i=1}^n \sum_{t=t_i+1}^{T_i} [N_{it} - N_{i,t-1}] \log(P_{it}) + \sum_{i=1}^n \sum_{t=t_i+1}^{T_i} [1 - N_{it} + N_{i,t-1}] \log(1 - P_{it}) \\ &= \sum_{i=1}^n \sum_{t=t_i+1}^{T_i} N_{it} \log(P_{it}) + [1 - N_{it}] \log(1 - P_{it}). \end{aligned}$$

The last line follows because $N_{it} - N_{i,t-1} = 1$ only when $N_{it} = 1$ since $N_{it} = 1$ implies that $N_{i,t-1} = 0$. This log-likelihood function is identical to the log-likelihood function of a regression model with binary dependent variables. Following the standard approach in bankruptcy modeling (e.g., Shumway [75]), we use the following logistic regression model to estimate the conditioned hazard rates

$$P_{it} = 1 / (1 + \exp(\beta X_{it})).$$

A firm contributes an observation to the regression model for every $t \in \{t_i, \dots, T_i\}$. The dependent variable N_{it} equals one if a bankruptcy filing is made by firm i in a particular year, zero otherwise. That is, if firm i declares bankruptcy, then $N_{it} = 0$ for $t \in \{t_i, \dots, T_i - 1\}$ and $N_{i,T_i} = 1$.

3.5.2 Explanatory Variables

In order to show the negative relation between inventory turnover and the probability of bankruptcy, we estimate the hazard rate with the private firm model's, Altman's [4], and Zmijewski's [85] variables with and without inventory turnover as an additional explanatory variable. These models are commonly used in the literature as a benchmark (e.g., Chava and Jarrow [21]). The purpose of our approach is to explore whether inventory turnover can increase the model fit in these three alternative models. The explanatory variables used in these models are:

- NITA (Net Income / Total Assets): A measure of the firm's profitability. Defined as Compustat item [NI] divided by Compustat item [AT]. Firms with high NITA should have lower probability of bankruptcy. It is an explanatory variable in the private firm and the Zmijewski models.
- TLTA (Total Liabilities / Total Assets): A measure of the firm's leverage. Defined as [LT/AT]. Firms with high TLTA should have higher probability of bankruptcy. It is an explanatory variable in the private firm and the Zmijewski models.
- WCTA (Working Capital / Total Assets): A measure of asset liquidity relative to total assets. Defined as ([ACT] minus [LCT]) divided by [ACT]. Firms with high WCTA should have lower probability of bankruptcy. It is an explanatory variable in the Altman model.
- RETA (Retained Earnings / Total Assets): A measure of cumulative profitability of a firm over its entire lifetime relative to total assets. Defined as [RE/AT]. Firms with high RETA should have lower probability of bankruptcy. It is an explanatory variable in the Altman model.

- EBTA (Earnings Before Interest and Taxes / Total Assets): A measure of the firm's profitability, excluding taxes and interest income/expense, relative to total assets. Defined as $[EBIT/TA]$. Firms with high EBTA should have lower probability of bankruptcy. It is an explanatory variable in the Altman model.
- METL (Market Value of Equity / Total Liabilities): According to Altman and Hotchkiss [5], this variable is a measure that shows how much the firm's assets can decline in value before the liabilities exceed the assets and the firm becomes insolvent. It is similar to option theoretic framework variables originating from Merton [60]. Defined as Common Shares Outstanding [CSHO] times the Closing Stock Price at the End of the Fiscal Year [prcc_f] divided by Total Liabilities [LT]. Firms with high METL should have lower probability of bankruptcy. It is an explanatory variable in the Altman model.
- SLTA (Sales / Total Assets): Measures the sales generating ability of the firm's assets. Defined as $[SALE/AT]$. Firms with high SLTA should have lower probability of bankruptcy. It is an explanatory variable in the Altman model.
- CACL (Current Assets / Current Liabilities): A measure of the firm's short term liquidity. Defined as $[ACT/ALT]$. Firms with high CACL should have lower probability of bankruptcy. It is an explanatory variable in the Zmijewski model.
- LNAGE (Natural logarithm of firm age): A measure of the firm's maturity. Defined as the natural logarithm of the difference between the current fiscal year and the fiscal year of the first financial statement that is available in Compustat plus one. (We add one to avoid $\ln(0)$ in the first firm-year

observation.) Some researchers argue that firms with high LNAGE should have lower probability of bankruptcy. Although it is not an explanatory variable in the original Altman z -model, it is usually added to the Altman model, e.g., Chava and Jarrow [21]. It is an explanatory variable in the Altman and the Zmijewski models we use in our paper.

In addition to these variables, we also use inventory turnover (IT) defined as Cost of Goods Sold [COGS] divided by the average Inventory [INVT] within a fiscal year. We use the fiscal year closing value of [COGS] and the average of the fiscal year opening and the fiscal year closing values of [INVT] to compute IT.

We present summary statistics for these independent variables in Table 3.4. Most variables vary in a wide range. For example, the average IT value is 6.61, but IT varies from a minimum of 1.15 to a maximum of 135.79. Despite this wide range, more than 98% of all IT values are less than 30. Hence, winsorization at the top and bottom 1% leads to a significantly narrower range. Figure 3.2 presents a histogram of inventory turnover values across all 4092 firm-year observations before winsorization.

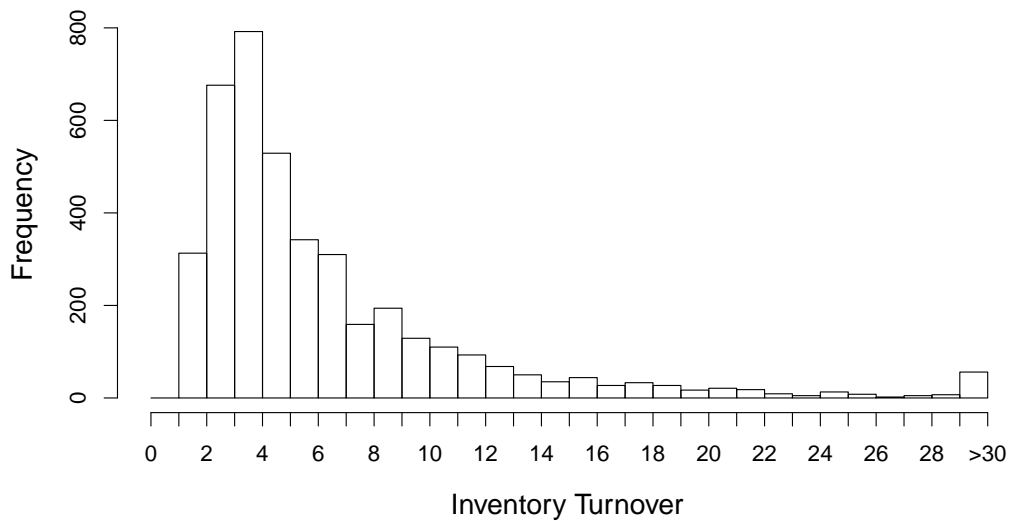
3.5.3 Regression Results

In this section, we present the logistic regression results for the private firm, Altman, and Zmijewski models with and without inventory turnover as an explanatory variable. The original models, which do not include inventory turnover as an explanatory variable, provide a sanity check regarding the statistical significance of previously used explanatory variables in our study, and

Table 3.4: Summary statistics of the independent variables before winsorization.

Variable	Median	Mean	Min	Max	Std. Dev.
IT	4.4044	6.6045	1.1533	135.7923	7.7185
WCTA	0.2466	0.2471	-0.1201	0.6189	0.1621
RETA	0.2632	0.2496	-0.6530	0.9004	0.2633
EBTA	0.1026	0.1076	-0.0810	0.3425	0.0754
METL	1.6081	2.9878	0.0505	24.5912	4.0042
SLTA	2.0194	2.2541	0.5158	5.3823	0.9835
NITA	0.0523	0.0487	-0.1975	0.2093	0.0657
TLTA	0.5473	0.5503	0.1271	1.3718	0.1996
CACL	1.8039	2.0774	0.3787	7.5237	1.0909
LNAGE	2.4850	2.4030	0.6931	3.9510	0.8450

Figure 3.2: Histogram of inventory turnover values across all firm-year observations before winsorization.



serve as a benchmark to measure the explanatory power of inventory turnover.

In the original private firm model, both net income to total assets (NITA) and total liabilities to total assets (TLTA) are statistically significant with the correct signs. That is, the conditional hazard rate increases in total liabilities to total assets, and decreases in net income to total assets. Adding inventory turnover (IT) as an explanatory variable to the original private firm model increases the model likelihood from 138.91 to 151.93. This increase is statistically significant

at $p \leq 0.001$. Hence, we conclude that adding inventory turnover as an explanatory variable increases the model fit. Inventory turnover is statistically significant at $p \leq 0.01$ with the correct sign, indicating that firms with high inventory turnover have lower probability of bankruptcy. Both net income to total assets and total liabilities to total assets remain statistically significant after inventory turnover is added to the model. Please see Table 3.5 for details.

Table 3.5: Logistic regression results with the private firm model variables.

Private Firm Model		
	Base Model	Base Model + IT
Intercept	-5.2302*** (0.4090)	-4.9652*** (0.4442)
IT		-0.1057** (0.0361)
NITA	-15.0577*** (1.5614)	-15.0284*** (1.6313)
TLTA	1.5223** (0.5519)	2.0354** (0.6296)
Model Fit:	138.92***	151.93***
Likelihood Ratio:		13.02***
Significance codes: *** : ≤ 0.001 , ** : ≤ 0.01 , * : ≤ 0.05 , · : ≤ 0.1		

In the original Altman model, working capital to total assets (WCTA), retained earnings to total assets (RETA), earnings before interest and taxes to total assets (EBTA), and market value of equity to total liabilities (METL) are statistically significant. The probability of bankruptcy decreases in each of these variables. However, sales to total assets (SLTA) and the natural logarithm of firm age (LNAGE) have no significance. Insignificance of firm age might arise due to our sample, which includes large retailers. Put differently, it might be significant due to mechanical reasons in a sample that includes small firms because small firms are more likely to fail within the first few years of their existence (Chava and Jarrow [21]). Adding inventory turnover as an explanatory vari-

able improves the model fit by increasing the likelihood from 154.84 to 166.36, which is a significant increase at $p \leq 0.001$. Inventory turnover has statistical significance with the correct sign at $p \leq 0.01$. That is, according to the Altman model, firms with high inventory turnover have lower bankruptcy risk. While retained earnings to total assets (RETA), earnings before interest and taxes to total assets (EBTA), and market value of equity to total liabilities (METL) remain statistically significant, working capital to total assets loses its significant after inventory turnover is added to the model. This might occur due to the high correlation between inventory and current assets. Please see Table 3.6 for details.

Table 3.6: Logistic regression results with the Altman model variables.

Altman Model		
	Base Model	Base Model + IT
Intercept	-2.2117*** (0.5842)	-1.9879** (0.6091)
IT		-0.1251** (0.0468)
WCTA	-0.2895*** (0.8648)	-1.4762 (0.9631)
RETA	-1.2743* (0.5944)	-1.6105** (0.6252)
EBTA	-15.7000*** (2.5290)	-15.0134*** (2.5331)
METL	-0.9463*** (0.2662)	-0.8589** (0.2741)
SLTA	-0.1459 (0.1521)	0.1392 (0.1899)
LNAGE	0.1240 (0.1826)	0.1449 (0.1855)
Model Fit:	154.84***	166.36***
Likelihood Ratio:		11.52***
Significance codes: *** : ≤ 0.001 , ** : ≤ 0.01 , * : ≤ 0.05 , · : ≤ 0.1		

Lastly, in the original Zmijewski model, net income to total assets (NITA)

and total liabilities to total assets (TLTA) are statistically significant, whereas the natural logarithm of firm age (LNAGE) and current assets to current liabilities (CACL) have no significance. Insignificance of current assets to current liabilities is surprising because current assets should capture the impact of inventory and current liabilities should capture inventory related liabilities (e.g., accounts payable) for a retailer. Once again, adding inventory turnover (IT) as an explanatory variable improves the model fit by increasing the likelihood from 139.52 to 152.37, which is a significant increase at $p \leq 0.001$. Inventory turnover has statistical significance with the correct sign at $p \leq 0.01$. After inventory turnover is added to the original Zmijewski model, net income to total assets and total liabilities to total assets remain statistically significant, and the natural logarithm of firm age and current assets to current liabilities continue to have no significance. Please see Table 3.7 for details.

Table 3.7: Logistic regression results with the Zmijewski model variables.

Zmijewski Model		
	Base Model	Base Model + IT
Intercept	-5.6124*** (0.6466)	-5.0136*** (0.6874)
IT		-0.10884** (0.0371)
NITA	-15.1629*** (1.5872)	-15.1491*** (1.6461)
TLTA	1.5713** (0.5657)	1.9980** (0.6389)
LNAGE	0.1070 (0.1688)	0.0889 (0.1703)
CACL	0.0494 (0.1317)	-0.0649 (0.1446)
Model Fit:	139.52***	152.37***
Likelihood Ratio:		12.85***
Significance codes: *** : ≤ 0.001 , ** : ≤ 0.01 , * : ≤ 0.05 , · : ≤ 0.1		

3.6 Limitations and Future Work

The analysis we present in the previous section is the first empirical step towards quantifying the impact of a retailer's operational performance on its bankruptcy risk. We show that adding inventory turnover as an explanatory variable to commonly used bankruptcy prediction models (i.e., the private firm model, the Altman model, and the Zmijewski model) improves model fit in all three cases and that retailers with low inventory turnover have higher probability of bankruptcy. Despite this promising result, we do not perform any out of sample forecasts due to the small sample size of our data set. Hence, we cannot claim that the prediction accuracy of our model is superior than that of the existing models.

Future work that will strengthen our message includes replicating our analysis on a larger data set including more bankruptcies and more industries, such as wholesalers and manufacturers. A larger data set will allow us to perform out of sample bankruptcy predictions and test whether IT improves prediction accuracy of the existing models. It will also enable us to perform long range forecasts. The prediction accuracy of an IT based model might be better than that of the existing models because inventory turnover might be a leading indicator for various organizational inefficiencies within the firm. Lastly, alternative explanatory variables can be used to find the best proxy for a firm's operational efficiency. Other operational metrics, such as fixed asset utilization, might have more explanatory power in industries that require higher fixed investments (e.g., production plants) than retailers.

BIBLIOGRAPHY

- [1] P. Aghion, T. Fally, and S. Scarpetta. Credit constraints as a barrier to the entry and post-entry growth of firms. *Economic Policy*, 22(52):731–779, 2007.
- [2] Y. Alan and V. Gaur. Operational investment and capital structure under asset based lending: A one period model. *SSRN eLibrary*, 2012.
- [3] Y. Alan, V. Gaur, and G. Gao. Does inventory turnover predict future stock returns? Available at <http://ssrn.com/paper=1971774>, 2011.
- [4] E.I. Altman. Financial ratios, discriminant analysis and the prediction of corporate bankruptcy. *Journal of Finance*, 23(4):589–609, 1968.
- [5] E.I. Altman and E. Hotchkiss. *Corporate financial distress and bankruptcy: predict and avoid bankruptcy, analyze and invest in distressed debt*. John Wiley & Sons Inc, 2006.
- [6] T.W. Archibald, L.C. Thomas, J.M. Betts, and R.B. Johnston. Should start-up companies be cautious? Inventory policies which maximise survival probabilities. *Management Science*, 48(9):1161–1174, 2002.
- [7] Asset Based Financing Journal. *BofA, others provide new revolver, term loan to Borders*, 04/01/2010.
- [8] V. Babich. Independence of capacity ordering and financial subsidies to risky suppliers. *Manufacturing & Service Operations Management*, 12(4):583–607, 2010.
- [9] V. Babich and M.J. Sobel. Pre-IPO operational and financial decisions. *Management Science*, 50(7):935–948, 2004.
- [10] T. Bates. Financing the development of urban minority communities: Lessons of history. *Economic Development Quarterly*, 14(3):227–242, 2000.
- [11] J. R. Birge and X. Xu. Firm profitability, inventory volatility, and capital structure. *SSRN eLibrary*, 2011.
- [12] Board of Governors of the Federal Reserve System. *Flow of funds accounts of the United States*. Washington, DC 20551, 2010.

- [13] O. Boyabatli and B. Toktay. Stochastic capacity investment and technology choice in imperfect capital markets. *Available at <http://ssrn.com/paper=1003551>*, 2010.
- [14] G. Burroughs. Asset-based loans can offer attractive financing option. *Puget Sound Business J*, 2008.
- [15] J. A. Buzacott and R. Q. Zhang. Inventory management with asset-based financing. *Management Science*, 50(9):1274–1292, 2004.
- [16] R. Caldentey and M. Haugh. Optimal control and hedging of operations in the presence of financial markets. *Mathematics of Operations Research*, 31(2):285–304, 2006.
- [17] J.Y. Campbell, J. Hilscher, and J. Szilagyi. In search of distress risk. *The Journal of Finance*, 63(6):2899–2939, 2008.
- [18] M. Campello and E. Giambona. Capital structure and the redeployability of tangible assets. *SSRN eLibrary*, 2010.
- [19] R.E. Carpenter and B.C. Petersen. Is the growth of small firms constrained by internal finance? *Review of Economics and Statistics*, 84(2):298–309, 2002.
- [20] X. Chao, J. Chen, and S. Wang. Dynamic inventory management with cash flow constraints. *Naval Research Logistics*, 55(8):758–768, 2008.
- [21] S. Chava and R.A. Jarrow. Bankruptcy prediction with industry effects. *Review of Finance*, 8(4):537, 2004.
- [22] P.D. Childs, D.C. Mauer, and S.H. Ott. Interactions of corporate financing and investment decisions: The effects of agency conflicts. *Journal of Financial Economics*, 76(3):667–690, 2005.
- [23] N.C. Churchill and J.W. Mullins. How fast can your company afford to grow? *Harvard Business Review*, 79(5):135–143, 2001.
- [24] G.L. Clementi and H.A. Hopenhayn. A theory of financing constraints and firm dynamics. *The Quarterly Journal of Economics*, 121(1):229–265, 2006.
- [25] Commercial Finance Association. *Annual asset-based lending and factoring surveys*, 2009.

- [26] M. Dada and Q. Hu. Financing newsvendor inventory. *Operations Research Letters*, 36(5):569–573, 2008.
- [27] AI Dimitras, SH Zanakis, and C. Zopounidis. A survey of business failures with an emphasis on prediction methods and industrial applications. *European Journal of Operational Research*, 90(3):487–513, 1996.
- [28] A. Dotan and S.A. Ravid. On the interaction of real and financial decisions of the firm under uncertainty. *Journal of Finance*, 40(2):501–517, 1985.
- [29] D. Duffie. *Measuring corporate default risk*. Oxford University Press, USA, 2011.
- [30] D. Duffie, L. Saita, and K. Wang. Multi-period corporate default prediction with stochastic covariates. *Journal of Financial Economics*, 83(3):635–665, 2007.
- [31] P.K. Dutta and R. Radner. Profit maximization and the market selection hypothesis. *Review of Economic Studies*, 66(4):769–798, 1999.
- [32] F.H. Easterbrook and D.R. Fischel. Limited liability and the corporation. *The University of Chicago Law Review*, 52(1):89–117, 1985.
- [33] R.A. Eisenbeis. Pitfalls in the application of discriminant analysis in business, finance, and economics. *The Journal of Finance*, 32(3):875–900, 1977.
- [34] E. Falkenstein, A. Boral, and L.V. Carty. Riskcalc for private companies: Moodys default model. *Moodys Investors Service, Global Credit Research*, 2000.
- [35] M.Z. Frank and V.K. Goyal. Trade-off and pecking order theories of debt. *Handbook of Corporate Finance: Empirical Corporate Finance*, 2:135–202, 2008.
- [36] GE Capital Commercial Finance, Inc. *Guide to asset based lending*, 1999.
- [37] D. C. Godfrey. SBA CAPLines: Taking the “risk” out of asset-based lending. *The Secured Lender*, 67(1):28, 2011.
- [38] M.J. Gordon. Corporate finance under the MM theorems. *Financial Management*, 18(2):19–28, 1989.

- [39] B. C. Greenwald and J. E. Stiglitz. *Macroeconomic Models with Equity and Credit Rationing*. University of Chicago Press, 1990.
- [40] M. Harris and A. Raviv. The theory of capital structure. *Journal of Finance*, 46(1):297–355, 1991.
- [41] T.H. Ho, S. Savin, and C. Terwiesch. Managing demand and sales dynamics in new product diffusion under supply constraint. *Management Science*, 48(2):187–206, 2002.
- [42] Q. Hu and M.J. Sobel. Capital structure and inventory management. *Unpublished manuscript*. Case Western Reserve University, 2005.
- [43] Q. Hu, M.J. Sobel, and D. Turcic. Optimization of inventory and dividends with risky debt. Available at <http://ssrn.com/paper=1592003>, 2010.
- [44] R.G. Hubbard. Capital-market imperfections and investment. *Journal of Economic Literature*, 36(1):193–225, 1998.
- [45] M.C. Jensen and W.H. Meckling. Theory of the firm: Managerial behavior, agency costs and ownership structure. *Journal of Financial Economics*, 3(4):305–360, 1976.
- [46] N.M. Kiefer. Economic duration data and hazard functions. *Journal of Economic Literature*, 26(2):646–679, 1988.
- [47] P. Kouvelis and W. Zhao. The newsvendor problem and price-only contract when bankruptcy costs exist. *Production and Operations Management*, 20(6):921–936, 2011.
- [48] A. Kraus and R.H. Litzenberger. A state-preference model of optimal financial leverage. *The Journal of Finance*, 28(4):pp. 911–922, 1973.
- [49] S. Kumar and J.M. Swaminathan. Diffusion of innovations under supply constraints. *Operations Research*, pages 866–879, 2003.
- [50] G. Lai, L.G. Debo, and K. Sycara. Sharing inventory risk in supply chain: The implication of financial constraint. *Omega*, 37(4):811–825, 2009.
- [51] M. T. Leary and J. R. Graham. A Review of empirical capital structure research and directions for the future. *SSRN eLibrary*, 2010.

- [52] M.T. Leary. Bank loan supply, lender choice, and corporate capital structure. *The Journal of Finance*, 64(3):1143–1185, 2009.
- [53] P.J. Lederer and V.R. Singhal. The effect of financing decisions on the choice of manufacturing technologies. *International Journal of Flexible Manufacturing Systems*, 6(4):333–360, 1994.
- [54] Legal News. *Final chapter. Liquidation brings end to Borders*, 09/05/2011.
- [55] M.L. Lemmon, M.R. Roberts, and J.F. Zender. Back to the beginning: Persistence and the cross-Section of corporate capital structure. *The Journal of Finance*, 63(4):1575–1608, 2008.
- [56] L. Li, M. Shubik, and M.J. Sobel. Control of dividends, capital subscriptions, and physical inventories. Available at <http://ssrn.com/paper=690864>, 2005.
- [57] L.M. LoPucki. UCLA bankruptcy research database, 2011.
- [58] P. MacKay and G.M. Phillips. How does industry affect firm financial structure? *Review of Financial Studies*, 18(4):1433–1466, 2005.
- [59] N.G. Mankiw. The allocation of credit and financial collapse. *The Quarterly Journal of Economics*, 101(3):455, 1986.
- [60] R.C. Merton. On the pricing of corporate debt: The risk structure of interest rates. *The Journal of Finance*, 29(2):449–470, 1974.
- [61] F. Modigliani and M.H. Miller. The cost of capital, corporation finance and the theory of investment. *The American Economic Review*, 48(3):261–297, 1958.
- [62] F. Modigliani and M.H. Miller. Corporate income taxes and the cost of capital: A correction. *The American Economic Review*, 53(3):433–443, 1963.
- [63] G.A. Moore. *Inside the tornado: Strategies for developing, leveraging, and surviving hypergrowth markets*. HarperBusiness, 2004.
- [64] S. C. Myers. Capital structure. *The Journal of Economic Perspectives*, 15(2):81–102, 2001.

- [65] S. C. Myers and N. S. Majluf. Corporate financing and investment decisions when firms have information that investors do not have. *Journal of Financial Economics*, 13(2):187 – 221, 1984.
- [66] S.C. Myers. The capital structure puzzle. *Journal of Finance*, 39(3):575–592, 1984.
- [67] J.A. Ohlson. Financial ratios and the probabilistic prediction of bankruptcy. *Journal of Accounting Research*, 18(1):109–131, 1980.
- [68] J. Ordover and A. Weiss. Information and the law: Evaluating legal restrictions on competitive contracts. *The American Economic Review*, 71(2):399–404, 1981.
- [69] Y.C. Park. The role of money in stabilization policy in developing countries. *Staff Papers-International Monetary Fund*, pages 379–418, 1973.
- [70] J. Peterman. The rise and fall of the J. Peterman Company. *Harvard Business Review*, 1999.
- [71] R. Radner and L. Shepp. Risk vs. profit potential: A model for corporate strategy. *Journal of Economic Dynamics and Control*, 20(8):1373–1393, 1996.
- [72] J. Rauh and A. Sufi. Explaining corporate capital structure: Product markets, leases, and asset similarity. Technical report, Northwestern University Working Paper, 2010.
- [73] M. Rothschild and J.E. Stiglitz. Increasing risk: I. A definition. *Journal of Economic Theory*, 2(3):225–243, 1970.
- [74] SBA. *Quick reference to SBA loan guaranty programs*, 2011.
- [75] T. Shumway. Forecasting bankruptcy more accurately: A simple hazard model. *Journal of Business*, 74(1):101–124, 2001.
- [76] J.E. Stiglitz. Some aspects of the pure theory of corporate finance: Bankruptcies and take-overs. *The Bell Journal of Economics and Management Science*, pages 458–482, 1972.
- [77] J.E. Stiglitz and A. Weiss. Credit rationing in markets with imperfect information. *The American Economic Review*, 71(3):393–410, 1981.

- [78] R. Swinney, G.P. Cachon, and S. Netessine. Capacity investment timing by start-ups and established firms in new markets. *Management Science*, 57(4):763, 2011.
- [79] R. Swinney and S. Netessine. Long-term contracts under the threat of supplier default. *Manufacturing & Service Operations Management*, 11(1):109–127, 2009.
- [80] The Secured Lender Industry News. *191 Community banks have received more than 2.4 billion from Treasury Department’s small business lending fund*, 2011.
- [81] J.A. Van Mieghem. Commissioned paper: Capacity management, investment, and hedging: Review and recent developments. *Manufacturing & Service Operations Management*, 5(4):269–302, 2003.
- [82] X. Xu and J.R. Birge. Joint production and financing decisions: Modeling and analysis. *Available at <http://ssrn.com/paper=652562>*, 2004.
- [83] X. Xu and J.R. Birge. Equity valuation, production, and financial planning: A stochastic programming approach. *Naval Research Logistics*, 53(7):641–655, 2006.
- [84] S. A. Yang and J. R. Birge. How inventory is (should be) financed: Trade credit in supply chains with demand uncertainty and costs of financial distress. *SSRN eLibrary*, 2011.
- [85] M.E. Zmijewski. Methodological issues related to the estimation of financial distress prediction models. *Journal of Accounting Research*, 22:59–82, 1984.